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The Influence and Modelling of Warping Restraint on Beams

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Introduction

Saint Venant, [1], established his theory of torsion (1853) by assuming axially invariant modes of tangential and axial (warping) displacements. In conjunction with known static boundary conditions the equations of elasticity were satisfied leading to the exact solution for pure torsion. His theory assumes free warping displacement and when this is restrained the torsional stiffness is increased to a degree dependent on the shape of the cross section. The basic beam finite element formulation assumes free warping but there are also elements that include a warping freedom thereby allowing warping to be controlled.

This article details a design scenario where the manufacturing process of a structural steel member of 'I' section was changed from rolling to machining. This change enabled thick integral end plates to be machined in to allow bolting to adjacent members. Prior to the design change warping restraint had not been considered but with the addition of integral end plates it became clear that a study would be required to establish how these restrained the warping and therefore changed the (torsional) stiffness of the member. Beam elements were used to model the structural members and the influence of different element formulations on the structural response are compared. In addition verified three dimensional solid models were used to provide validation for the beam solutions. To verify the modelling approach adopted and to provide solutions that may be checked with closed-form solutions, members with other cross sections are also considered.

In preparing this article benchmark studies on warping restraint were not found even in the documentation of ANSYS, the suite of FE software used for this study. It is hoped therefore that this article might be useful to fellow structural analysts when considering how to model beams with warping restraint.

Geometry and Material Properties

The three cross sections considered are shown in figure 1.

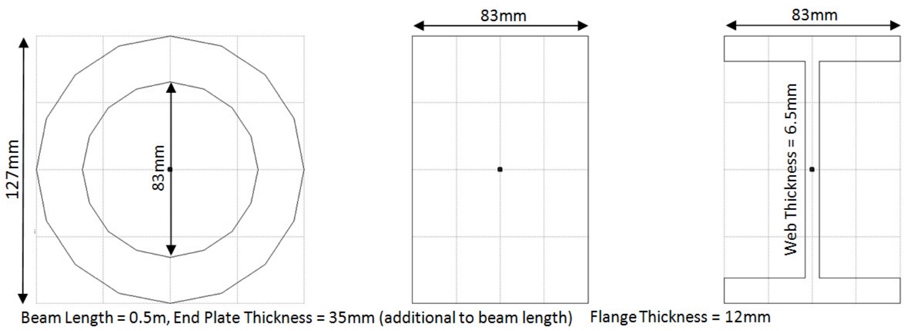


Figure 1: Cross Sectional and other Geometric and Material Properties

Closed-Form Solutions

The theory of pure torsion defines the (torsional) stiffness of a beam of length, L, as the torque, T, divided by the relative rotation, θ, of the two ends of the beam measured in radians:

$$\frac{T}{\theta} = \frac{GJ}{L}$$

where G is the shear modulus and J is the polar second moment of area of the cross section. The stiffnesses (reported in this article in kNm/rad) for the three beams defined above may be calculated, [2], and the values are 3293, 2269 and 16.6 respectively for the circular, rectangular and 'I' section beams.

Boundary Conditions

For pure torsion the end sections of the member, where the torque is transferred in and out of the member, are assumed to rotate such that the tangential displacement is proportional to the distance from the axis of rotation. The axial displacement at the end sections depends on the warping restraint:

1) **Free Condition** – nodes on the end sections are free to move independently in the axial direction.
2) **Restrained Condition** – nodes on the end section remain in the same plane which is free to translate axially (although as a result of symmetry this translation will be zero).

To implement these kinematic conditions one first needs to recognise that nodes of solid elements possess only translational freedoms. In order to transfer rotation one needs to introduce nodes with rotational freedoms and the simplest way to do this is to add a beam element extending each end section outwards and lying on the centroidal axis of the member.

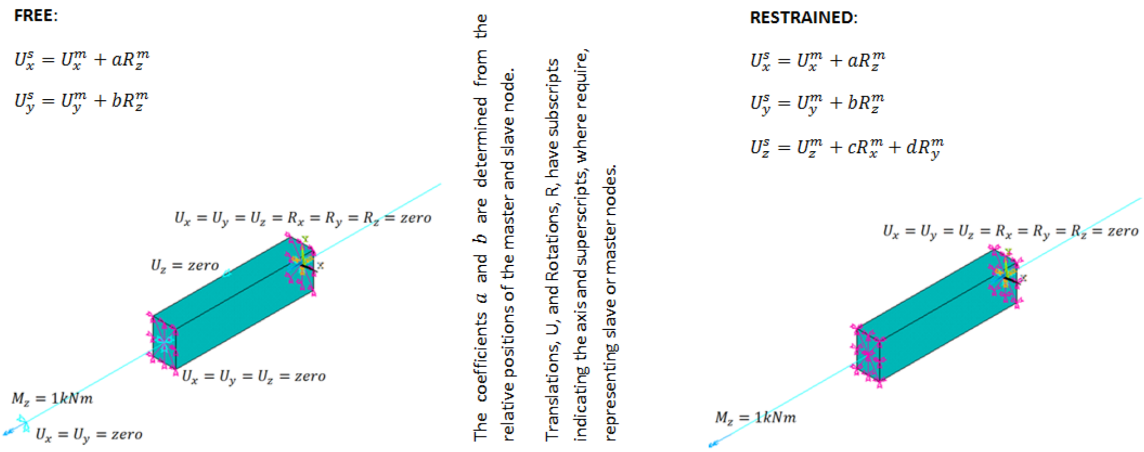


Figure 2: Boundary Conditions

The nodes at the ends of the beam lying in the plane of the end sections are created distinct from the nodes of the solid model and coupled using the CERIG function in ANSYS. In this manner the correct constraint equations are written between the freedoms of the beam element (master) node (including rotations) and the translations of the slave nodes on the end plane of the solid model.

The model also needs single point constraints to remove any rigid-body motions and to deal with the incomplete coupling. The model is driven with a 1kNm torque applied to the node at the left-hand end of the left-hand beam. The boundary conditions are illustrated in figure 2.

Finite Element Models

Solid models were constructed using twenty-node SOLID186 reduced integration brick elements and the level of mesh refinement is indicated in figure 3. For the beam models meshes of BEAM188 elements were used with default Key Options. The number of elements used for the main member was 100 and 20 elements were used for each end plate. The section properties of the beam were defined as per figure 1 with a step transition in properties at the junction between 'I' beam and end plates. These models have been verified in terms of mesh convergence to produce stiffness values within 1% of the converged value [3].

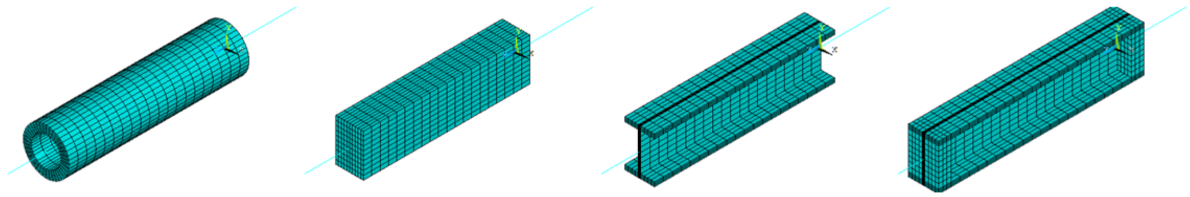


Figure 3: Solid Finite Element Models

Results for Solid Models

Table 1 illustrates, qualitatively, what an engineer already knows namely that axisymmetric sections (circular) do not warp, closed-sections (rectangular) do warp but that the degree of warping is significantly less than open-sections ('I' sections). The values for free warping agree well (exactly for the circular and rectangular sections) with the closed-form values already presented.

Shape	Type	Free	Restrained	% Increase
Circular	Axisymmetric	3293	3293	0.0
Rectangular	Closed	2270	2338	3.0
'I' Section	Open	16	87	451.8

Table 1: Stiffnesses for Three Sections

Given the degree to which restraining the warping of the 'I' beam increases the stiffness it is not surprising to see (table 2) that the addition of integral end plates will have a similar but partial effect and for this example the stiffness is increased by a factor of nearly three over the standard 'I' beam.

End Plates	Free	Restrained	% Increase
No	16	87	451.8
Yes	58	72	24.2

Table 2: Stiffnesses for 'I' Section with and without End Plates

Figure 4 shows contours of axial displacement together with the maximum axial displacement in micrometers rounded up to the nearest whole value. Symmetric contour ranges were chosen with red indicating +ve displacement and blue –ve displacements. For the uniform members (those without end plates) with unrestrained warping the axial displacement is invariant with axial position and show the classical warping distribution with opposite signs at adjacent corners of the section. When warping is restrained, away from the ends of the member warping still occurs but has to transition to zero at the ends of the member.

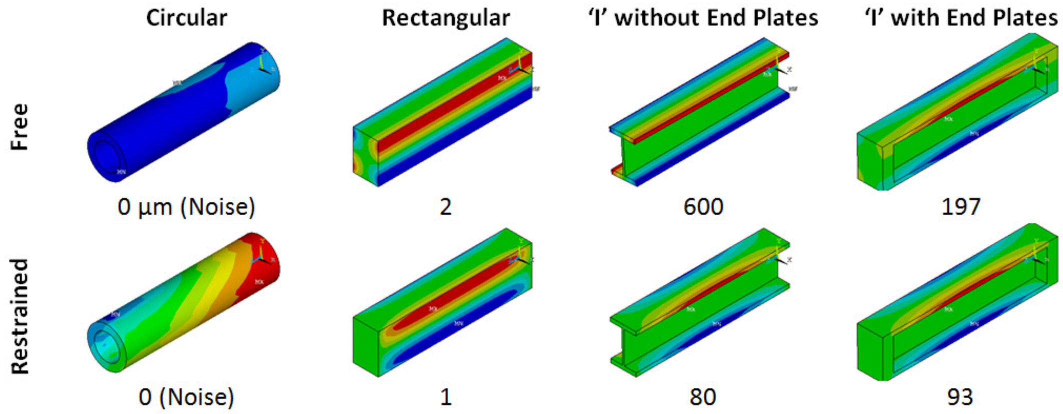


Figure 4: Contour Plots of Axial Displacement (maximum values in µm)

Results for Beam Models

BEAM188 has two formulations; one which does not explicitly include warping and one which does. For the formulation that includes warping an additional warping freedom is added to each node. An extract from the ANSYS Help Manual is shown in figure 5 together with the corresponding dialogue box for setting the element's Key Options.

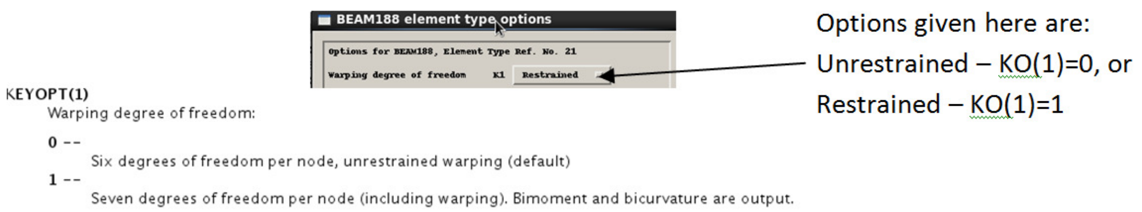


Figure 5: BEAM188 Key Options

To aid understanding of the beam formulation the following definitions are adopted:

KO(1)=0 – standard (default) formulation without warping freedoms

KO(1)=1 - formulation with warping freedoms:

For KO(1)=1 the nodes have additional warping freedoms which may be unrestrained or restrained:

KO(1)=1f – formulation with warping freedom unrestrained

KO(1)=1r – formulation with warping freedom restrained at the ends of the model

The default formulation in ANSYS is KO(1)=0 but for beam sections that are deemed open ANSYS provides a warning that the user should consider using KO(1)=1 presumably with appropriate constraining of the warping freedom. Table 3 lists the stiffnesses for the beams with the figure in brackets being the percentage increase over the values obtained for the corresponding solid model.

Shape	Free KO(1)=0 & KO(1)=1f	Restrained KO(0)=1r	% Increase
Circular	3286 (0)	3319 (1)	1.0
Rectangular	2308 (2)	2422 (4)	5.0
'I' Section	17 (5)	89 (3)	439.8

Table 3: Stiffnesses for Three Sections (Beam Elements)

Whilst there is a difference between the stiffnesses of the beam and solid models, with the beam models tending to be stiffer than the solid model, the results are consistent and, with no greater than a 5% difference can be, considered to be a reasonable engineering approximation.

The results for KO(1)=0 and KO(1)=1f are identical and this reminds us that the formulation without explicit warping (KO(1)=0) actually models free warping. It is also seen that the ANSYS dialogue box is misleading since with KO(1)=1 the warping remains unrestrained until the user changes the default free warping freedoms.

The results for the 'I' section are compared in table 4. Again the numbers in brackets are the percentage change in stiffness compared with the solid model. Note, however, that for the beam with end plates the beam model is now less stiff than the solid model.

End Plates	Free KO(1)=0	Free KO(1)=1f	Restrained KO(1)=1r
No	17 (5)	17 (5)	89 (3)
Yes	17 (-241)	57 (-2)	60 (-20)

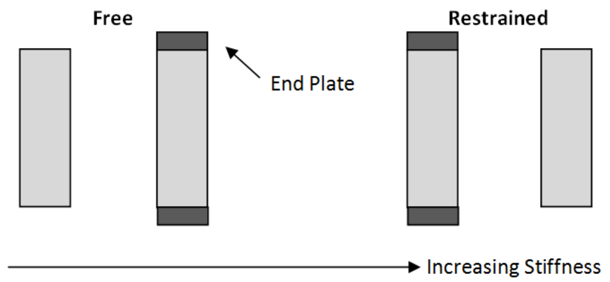
Table 4: Stiffnesses for 'I' Section with and without End Plates (Beam Elements)

The second row of table 4 is for the member including end plates and will now be discussed. The first point to note is the massive discrepancy between the results for the KO(1)=0 beam model and the solid model with free warping. The explanation for this is that this beam formulation does not ensure continuity of warping between beams (there are no warping freedoms). As such the partial restrain on the warping expected (and seen for the solid model) is not captured. The beam formulation KO(1)=1f offers a far more realistic solution being only 2% less than the result for the solid model. For the beam with restrained warping (KO(1)=1r) the stiffness increases but significantly less than that for the solid model and the stiffness is underestimated by some 20%.

Closure

The results for the 'I' beam are summarised in figure 6 where it is seen that the basic beam element, without warping control, is clearly unsuitable for modelling situations where warping is partially or fully constrained. The more advanced element, which includes warping control, performs significantly better particularly when end plates are not included. When end plates are included then the advanced beam model can lead to error, c.f. the restrained warping case. But for the geometry considered in this article the free warping case produces good correlation with the solid model.

The machined member in this study is to be bolted to thick members and so it is likely that warping at the member ends would be almost completely restrained by the adjacent structure [4]. As such, if the member had been modelled with beam elements without warping restraint then the stiffness would have been underestimated by some 72/16=4.5 times! Although when warping is restrained the beam model still underestimates the stiffness but by only 72/60=1.2 times.



Solid	16	58	72	87
Beam	17	57	60	89

Figure 6: Summary of Stiffnesses for the 'I' Section

The error in the results shown above for beam elements reminds us of the importance of considering carefully the appropriateness of the choice of idealisation. The sort of study here presented is thus necessary if the engineer is to make a sound, evidence-based, decision as to the nature of the idealisation to choose. An alternative possible approach, which may have both simplified the analysis and led to more reliable results, would have been to replace the 'I' sections with circular sections for which warping would not have been an issue. This indicates the potential virtue of adopting a 'design-for-analysis' philosophy which, particularly for one-off structures, has many potential virtues.

The absence of suitable benchmark verification problems for warping in beam finite elements provided part of the motivation for writing this article. In this study it was found that the dialogue for setting the element Key Options in ANSYS was highly misleading in that it suggests that warping is restrained when in fact it is not without further user intervention. The default option for BEAM188 is KO(1)=0. This, however, is only appropriate for axisymmetric cross sections and as such a more appropriate default might be KO(1)=1 which, of course, should also cater for non-warping axisymmetric cross sections. The following recommendations are therefore suggested to ANSYS Inc:

1) Change the dialogue text from 'Restrained' to 'Included',
2) Add some benchmark examples and advice to the Help Manual, and,
3) Change the default value for KO(1) from 0 to 1.

References

[1] Timoshenko, S.P., ***History of Strength of Materials***, Dover (1983) – ISBN 0-486-61187-6

[2] Hearn, E.J., ***Mechanics of Materials***, Pergamon (1985) – ISBN 0-08-030529-6

[3] http://www.ramsay-maunder.co.uk/downloads/warping_article_web.pdf

[4] Hicks, S., ***Design of Members Subjected to Torsion***, AD249, Steel Construction Institute

APPENDIX 1: Verification

The results presented in this article and the conclusions that are drawn from these are of course only valuable if the quantities of interest have been established accurately. This is a question of verification of the FE results. In this context mesh refinement studies have been performed for both beam and solid models of the 'I' beam with end plates and restrained warping.

As both beam and solid elements are displacement type elements and the problem is 'force' driven then one can expect an integral quantity such as the torsional stiffness to converge monotonically from above the true value. Note also that as the quantity of interest is based on the displacement response of the model, accurate results will be achieved with a lower level of mesh refinement than would be required for stress results.

Beam Element Model

The minimal mesh for this problem is one beam for each of the end plates and the central beam. The torsional stiffness is recorded for this minimal mesh and uniformly refined meshes up to and including 64 elements per section.

1	110.011
2	65.445
4	61.013
8	60.058
16	59.829
32	59.771
64	59.758

Table 5: Torsional Stiffnesses

The results show convergence (from above) up to three significant figures and the percentage error in the 20x100x20 mesh used in this report is 100%*(60.195-59.758)/60.195= 0.7%.

Solid Element Model

Results for a uniform refinement and a uniform derefinement of the mesh already used are reported below:

2	72.37
1	71.99
0.5	71.77

$$K = \frac{2^n K(\frac{h}{2}) - K(h)}{2^n - 1}$$

Table 6: Torsional Stiffnesses

Richardson's Extrapolation is used to estimate an exact solution based on a pessimistically chosen unit convergence rate we thus obtain 71.56 which gives an error 100%*(71.99-71.56)/71.99= 0.6%.

With both beam and solid models producing errors less than 1% we will consider the results presented in this article to be suitably verified.

APPENDIX 2: Contour Plots

Contour plots of stresses and displacements for the four sections considered are shown in figures 7 – 10.

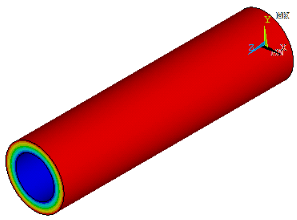
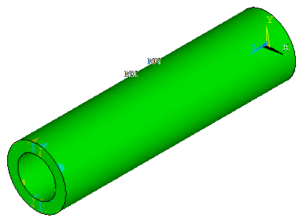
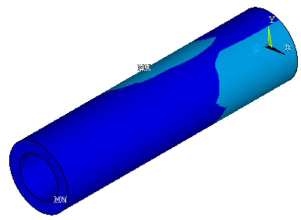
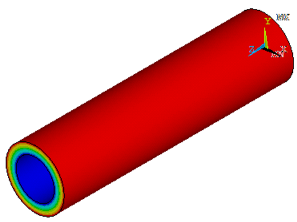
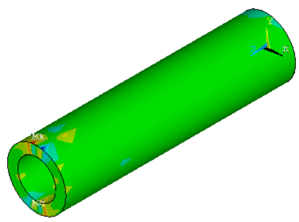
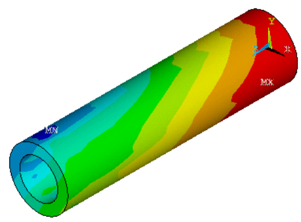
Warping	**Mises Stress**	**Axial Stress**	**Axial Displacement (mm)**
Free			
	5.3MPa	Noise	Noise
Restrained			
	5.3MPa	Noise	Noise

Figure 7: Contour Plots for Circular Section

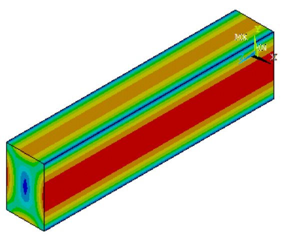
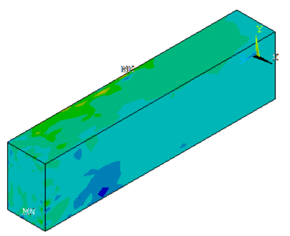
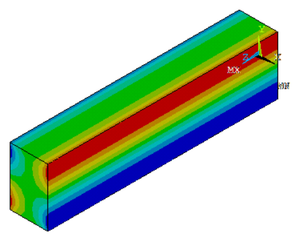
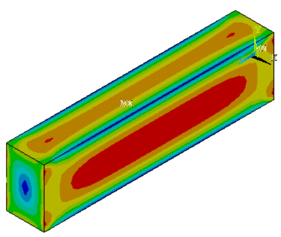
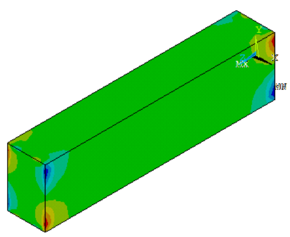
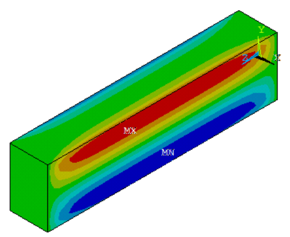
Warping	**Mises Stress**	**Axial Stress**	**Axial Displacement (mm)**
Free			
	8.6MPa	Noise	1.6e-3
Restrained			
	9.0MPa	9.0MPa	0.9e-3

Figure 8: Contour Plots for Rectangular Section

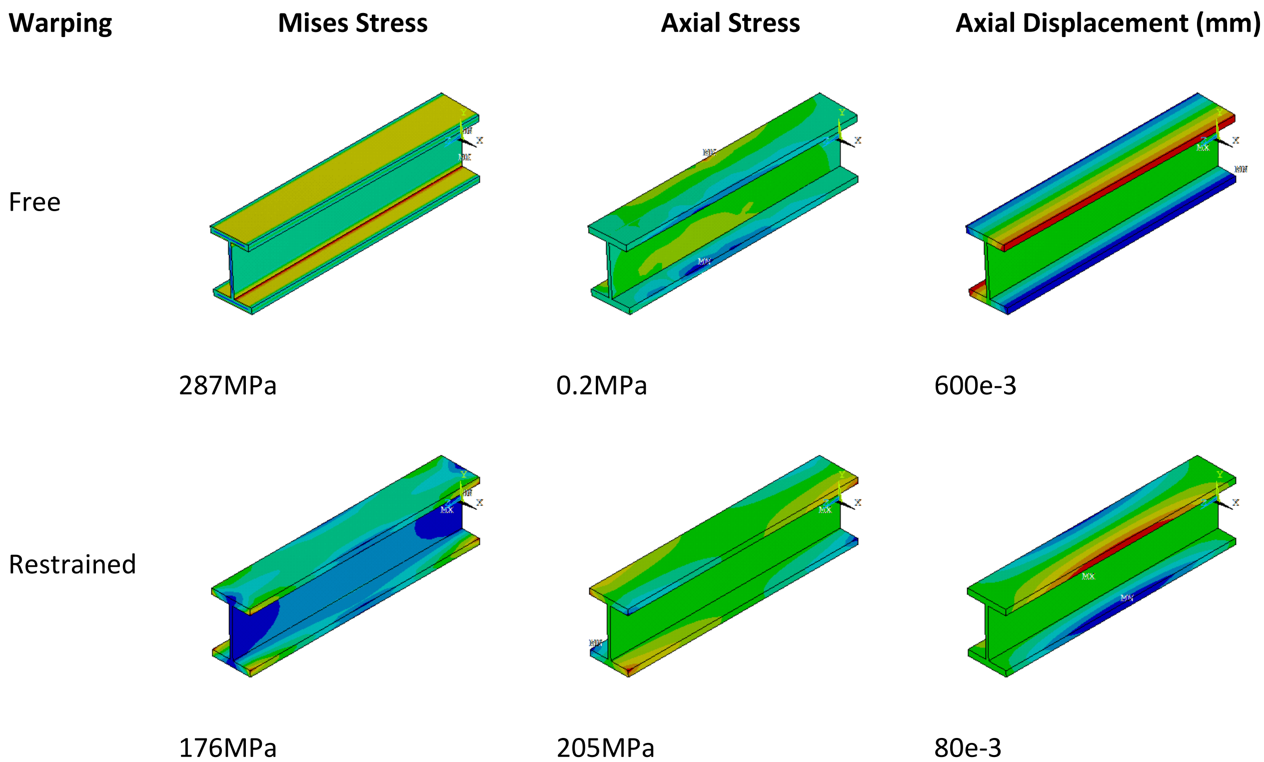


Figure 9: Contour Plots for 'I' Section

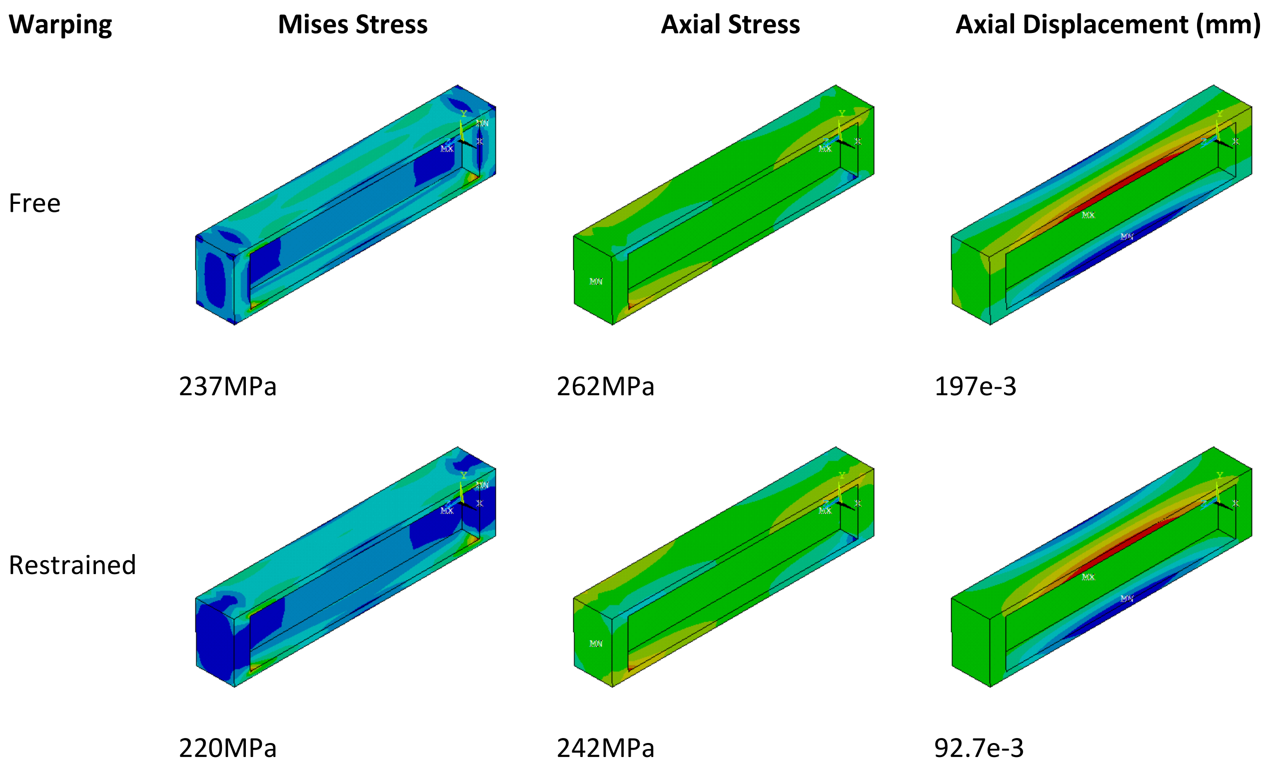


Figure 10: Contour Plots for 'I' Section with Integral End Plates

APPENDIX 3: Verification of Section Rotation

The kinematic coupling between the end node of the beam and the nodes on the end of the solid model assume a 'strength of materials' relationship where the end plane of the solid model rotates as a rigid-plane, i.e. all nodes undergo the same rotation. This assumption may be verified by confirming that this relationship also holds away from the ends of the solid model where Saint Venant's principle would lead to displacements that are not affected by the kinematic coupling. To this end rotation of nodes on a near central plane of the solid 'I' section model (unrestrained warping) have been evaluated according to figure 11.

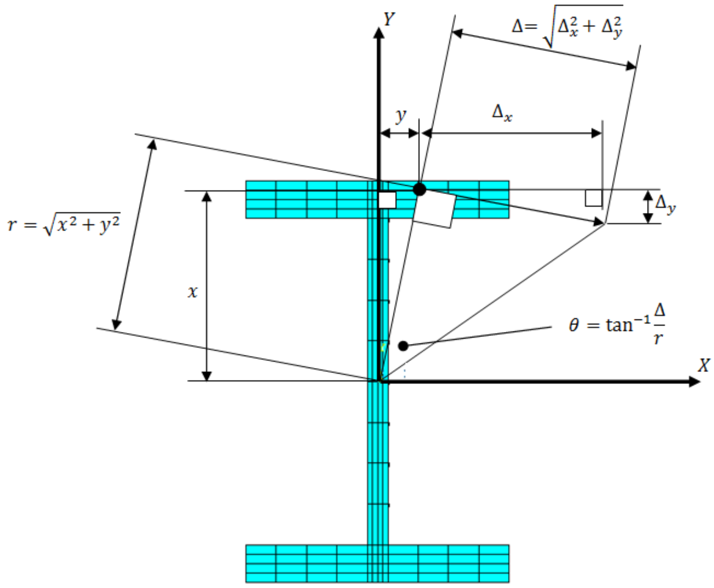


Figure 11: Calculation of rotation for a node

This exercise showed that the extremal values of the rotation (theta) were within 0.005% of each other and demonstrates that the assumed kinematic coupling is indeed correct.