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# Supplement to Letter to Verulam

## Background

RMA has developed equilibrium finite element software (EFE) for the elastic and plastic design and assessment of, amongst others, reinforced concrete slabs and bridge decks. The ongoing Verulam discussion on Effective Width of Slabs was of interest to us since, with the safe plastic analysis techniques available within EFE, the calculation of effective widths, albeit currently assuming adequate ductility, is simply conducted. We submitted a letter to The Structural Engineer summarising the results obtained from EFE on a particular slab configuration discussed in the letter. Here we present supplementary results which, for reasons of space, did not go into the letter.

## Elastic Solution

The slab configuration considered in Verulam is a 12m by 6m one-way (short dimension) spanning simply supported slab with central point load. A 6m by 3m symmetric quadrant of the slab was modelled as shown in figure 1. The elastic properties and thickness are given in the figure together with the boundary conditions (symmetry on two edges and simple support on one edge) and the loading (25kN on one quadrant distributed evenly over a 0.1m by 0.1m region at the centre of the plate). The simple support condition that we model is 'hard', in the context of Reissner-Mindlin plate theory, i.e. torsional moments form part of the reactions.

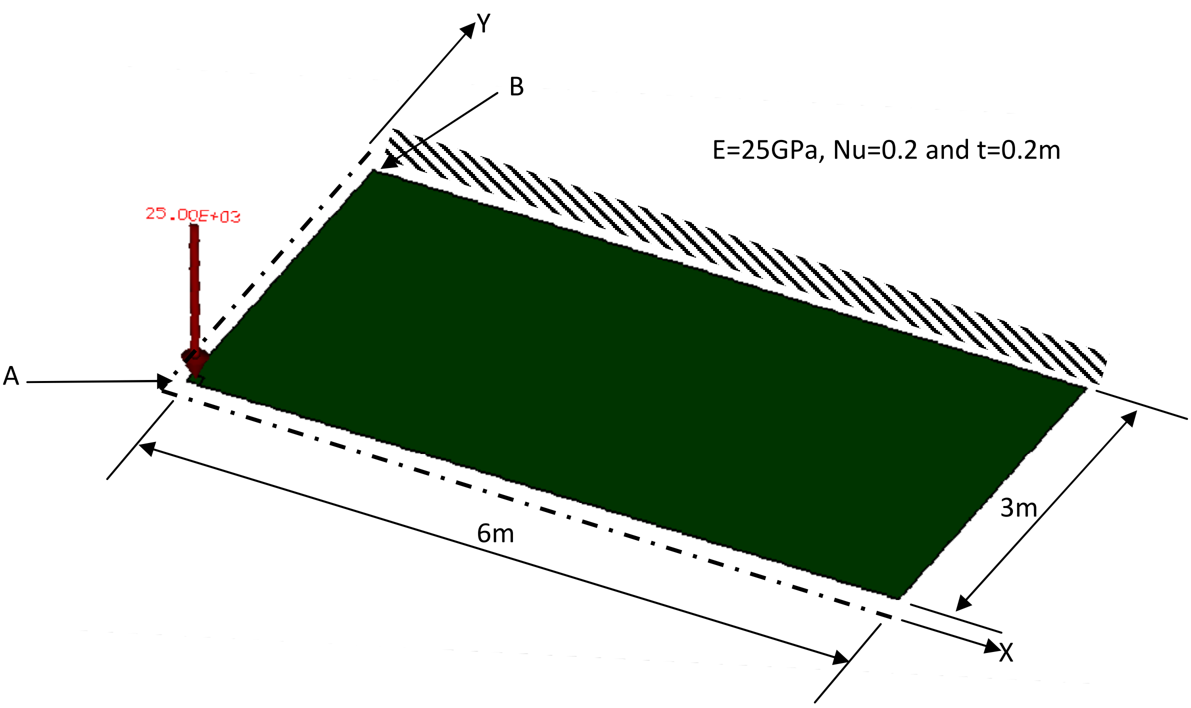
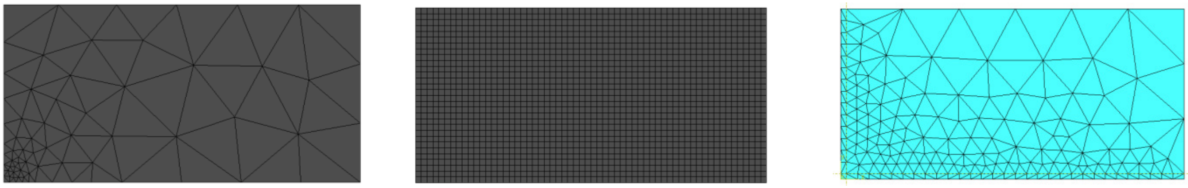



**Figure 1:** Geometry, material, boundary conditions and loading

A mesh refinement study using the two meshes shown in figure 2(a) and (b) was conducted with moment fields varying from quadratic to quartic (degree 2 to 4).



(a) 112 triangles (EFE) (b) 1800 squares (EFE, OASYS) (c) 347 triangles (ABAQUS)

**Figure 2:** Finite element meshes

Three quantities of interest were monitored for convergence these being the transverse displacement at point A, the moment Myy at point A and the shear Qy at point B. The results are shown in table 1 which also includes FE results from ABAQUS and OASYS (both programs use conventional conforming elements), Bill Wadsworth's finite different results (BW) and Robert Hairsine's grillage results (RH). Note that RH's results have been inferred from his letter (Verulam, 16th November 2010) where he states that his results were within 10% and 5% of BW's results respectively for moments and shears – we have assumed that the results take him nearer to the correct value.

| | Number of Elements | Degree of Moment (M) or Displacement (D) | Uz (mm) | Myy (kNm/m) | Qy (kN) | |
|---|---|---|---|---|---|---|
| EFE | 112 | 2M | 3.66 | 41.90 | 8.61 | |
| | 112 | 3M | 3.66 | 41.94 | 8.46 | |
| | 112 | 4M | 3.66 | 41.94 | 8.45 | Converged |
| | 1800 | 2M | 3.66 | 41.94 | 8.45 | Results |
| ABAQUS | 347 | 1D | 3.61 | 38.59 | | |
| | 347 | 2D | 3.66 | 43.58 | | |
| OASYS | 1800 | 1D | | 39.44 | 8.44 | |
| BW | Finite Difference | | | 33.62 | 12.64 | |
| RH | Grillage | | | 36.98 | 12.01 | |

**Table 1:** Convergence of quantities of interest with mesh refinement

The mesh refinement study indicates that the results obtained for the 112 mesh with quartic moment fields have converged as they are identical to the much more refined 1800 element mesh. The conventional conforming finite element models agree well with EFE when quadratic displacement fields are used – the results for the linear displacement elements are, as expected, less accurate.

It is interesting to note how different the finite difference and grillage results are from the true values – 20% underestimate for moment and 42% overestimate for shear. It is interesting also to see how good the results from EFE are for the coarse model.

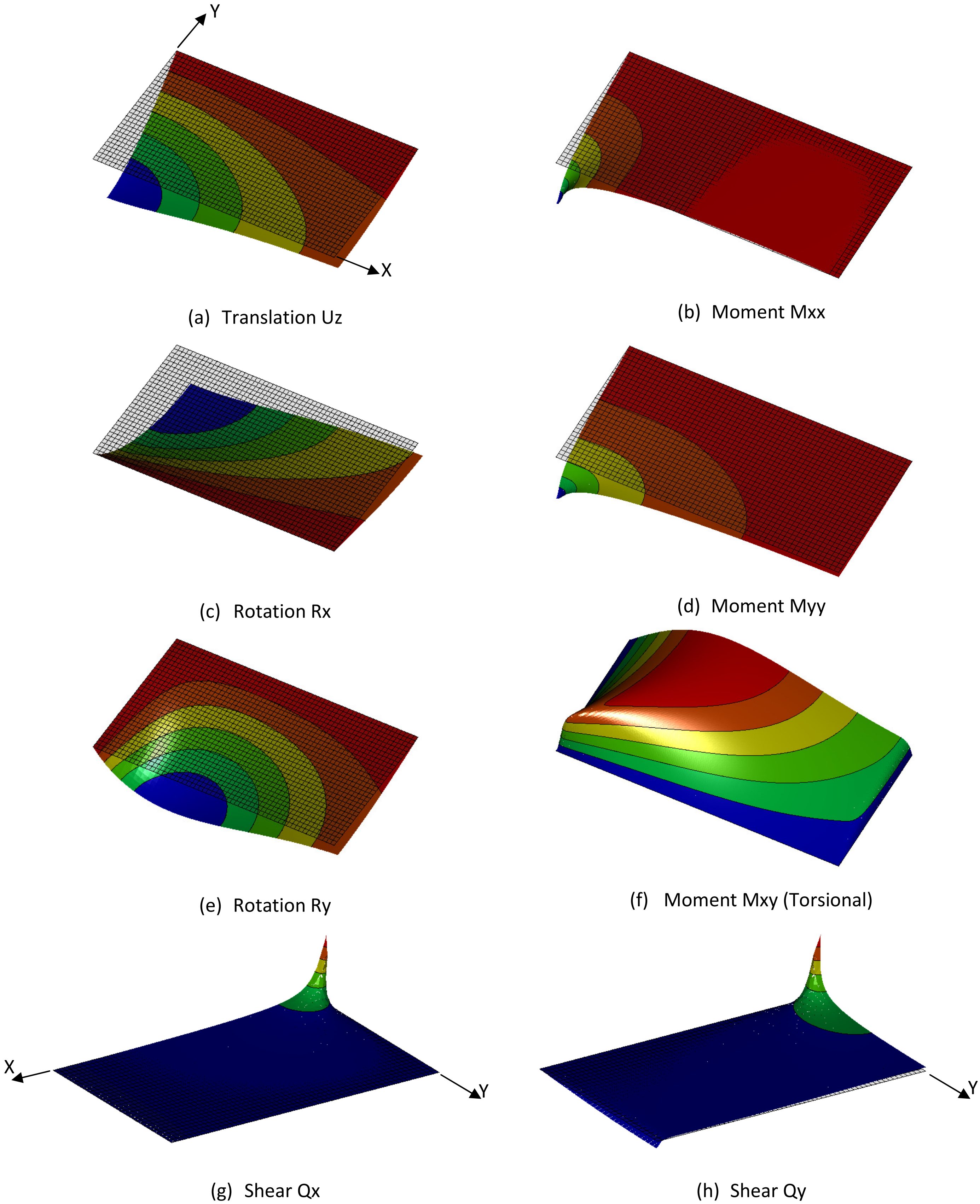
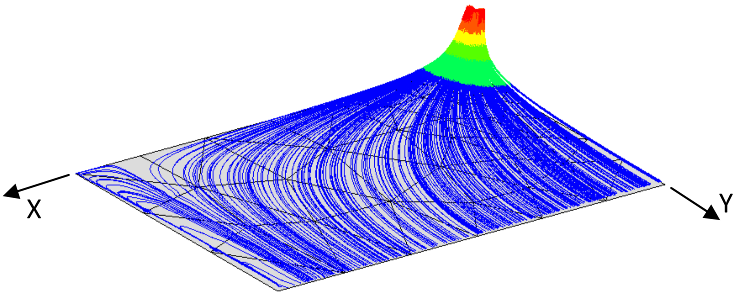
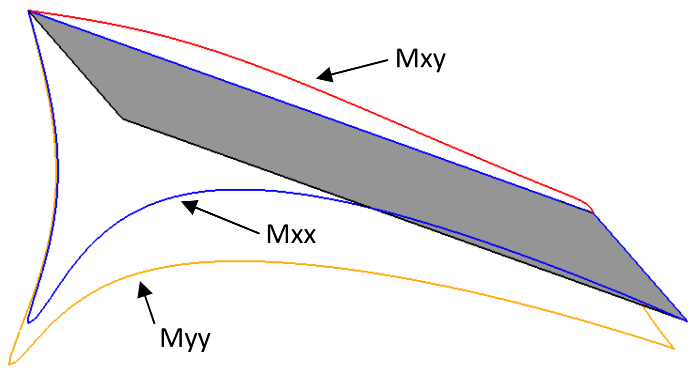



**Figure 3**: Contour plots of the displacements, Cartesian moments and shears

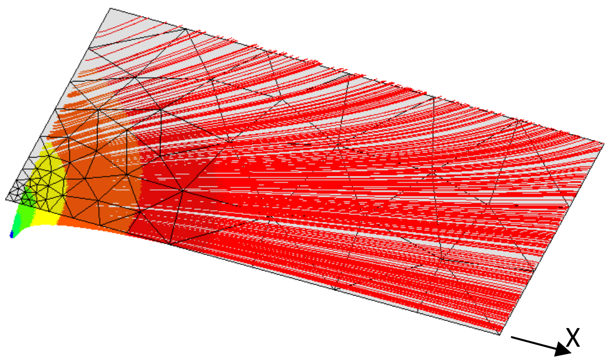
Contour plots of the displacements, Cartesian moments and shears are shown in figure 3. In the moment plots, hogging moments are positive and are plotted above the plane of the elements, sagging moments are negative and are plotted below. Note that these are unprocessed results, i.e. they are plots of the moments and shears from the finite element model. Unlike conforming finite elements these quantities are in equilibrium with the applied load and conform with the static boundary conditions – for example Mxx and Myy should be zero on the simply supported and free edges and Mxy should be zero on all except the simply supported edge where torsional moments were restrained (hard simple support).



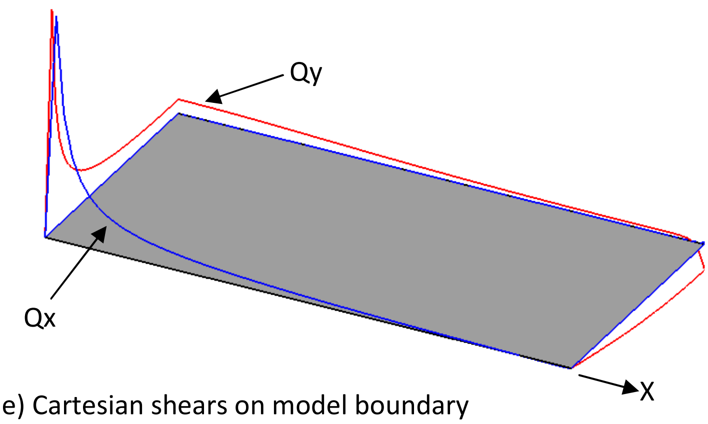

(a) Resultant shear trajectories



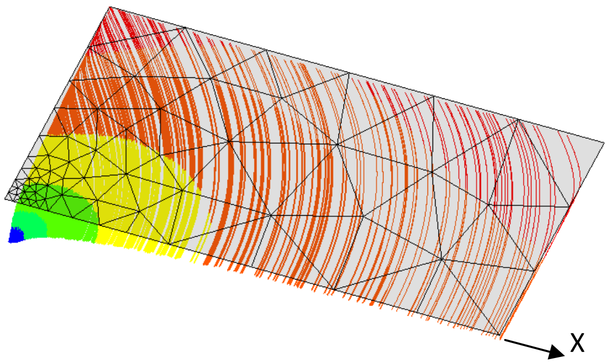

(d) Cartesian moments on model boundary




(b) Maximum principal moment trajectories




(e) Cartesian shears on model boundary




(c) Minimum principal moment trajectories

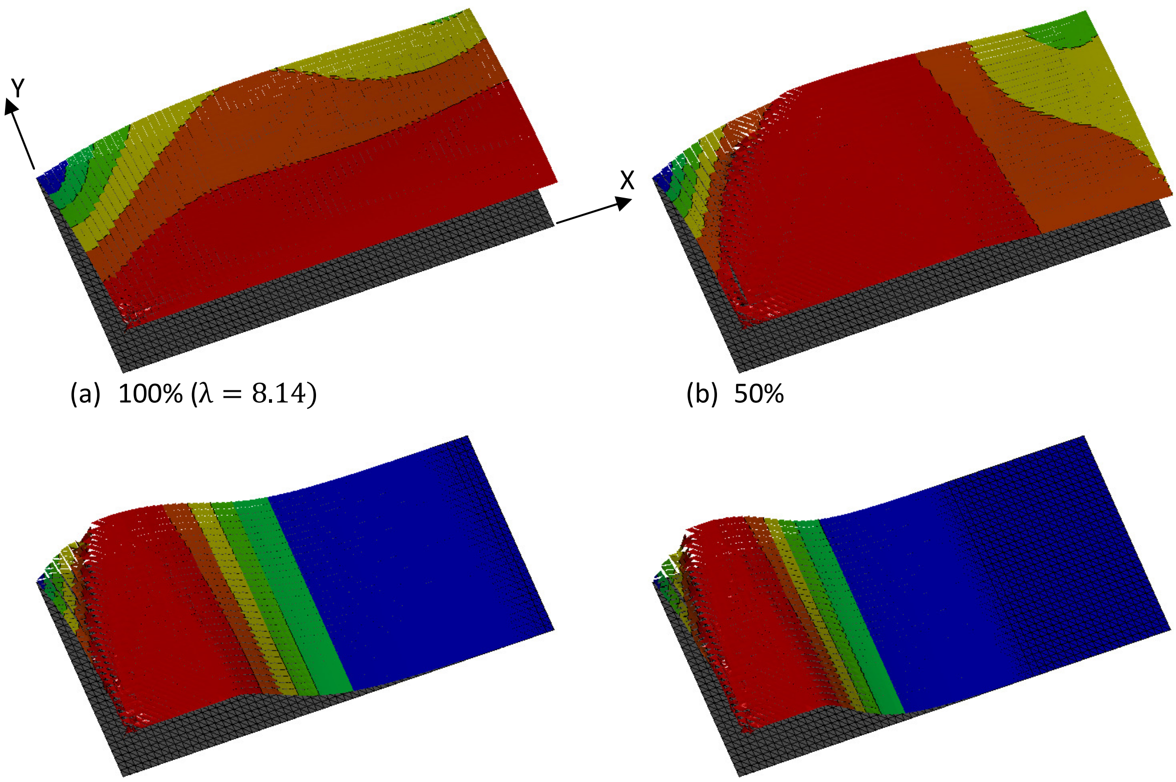
**Figure 4:** Plots of shear and moment trajectories and boundary distributions of moments and shear

One of the virtues of EFE is that, with equilibrium being satisfied *a-priori,* high quality results of practical engineering significance are immediately available. Figure 4 shows some of these results including trajectories, which aid understanding of the way in which the load is transmitted through a structure, and boundary distributions which illustrate how the load is transferred into adjacent structures.

## Plastic Solution

In addition to elastic analyses, EFE performs plastic ULS analysis of, amongst others, reinforced concrete plates. The moment fields used are in equilibrium with the applied load and the Nielsen bi-conic yield criterion (or alternatively the Wood-Armer yield criterion) limits the values of the moments. The scheme is a rigorous lower-bound approach providing guaranteed safe, conservative, estimates of the flexural collapse load (when shear is not critical) irrespective of mesh refinement. The moment fields are constructed for the plastic solution based on Kirchhoff type elements which enforce continuity of bending moments and equivalent Kirchhoff shear forces.

The software also includes a conventional yield-line solver for obtaining traditional upper-bound solutions for comparison purposes. We have conducted yield line analyses for cases with yield moments of 100kNm/m for both hogging and sagging in the span direction, and with transverse yield moments at 100% (isotropic), 50%, 10% and 5% of this value. For the isotropic case, upper and lower bound solutions agree at a load factor (λ) of 8.14, and as the transverse yield moment is reduced so is the load carrying capacity. Figure 5 shows contours of utilisation for the four transverse yield moments considered.




(a) 100% ($\lambda = 8.14$)

(b) 50%

(c) 10% ($\lambda = 3.37$)

(d) 5%

Figure 5: Utilitisation for various percentages of transverse yield moment (EFE)

Figure 6 shows the yield line collapse mechanism with a single geometric variable X. In this figure the blue line represents a sagging yield line and the dashed red line a hogging yield line. This mechanism is a simplified first approximation of the true collapse mechanism which in practice will probably be more complicated. The load factor from the refined EFE model is probably within a few percent of the true value and the inset to figure 6 shows how both upper and lower bound load factors vary with the geometric variable X for the eight element mesh. It is seen that whereas the yield line solution is extremely sensitive to the value of X, the lower bound solution from EFE remains sensibly invariant despite an extremely coarse mesh.

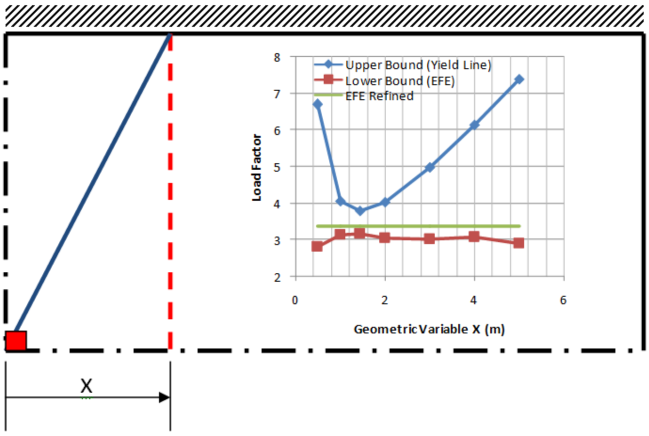



**Figure 6**: Geometric Optimisation for Yield Line (10% transverse yield moment)

Boundary distributions of moment are shown in figure 7 for the case of 10% transverse yield moment. It should be noted that in this figure the torsional moment Mxy is not exactly zero along the lines of symmetry, particularly in the neighbourhood of the load, this being a consequence of the use of Kirchhoff type elements.

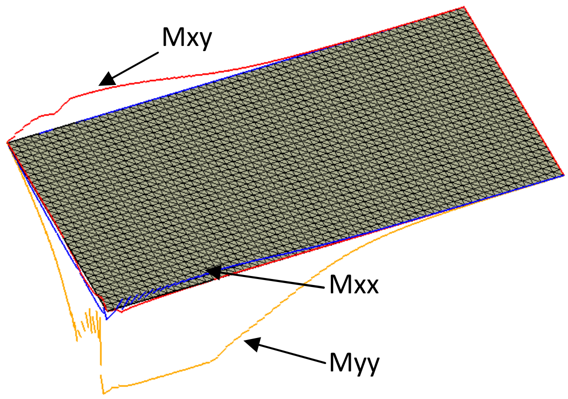



**Figure 7:** Boundary distributions for EFE (10% transverse yield moment)

In figure 8 the boundary distributions of bending moments along the centre line of the slab for the various analyses conducted are shown.

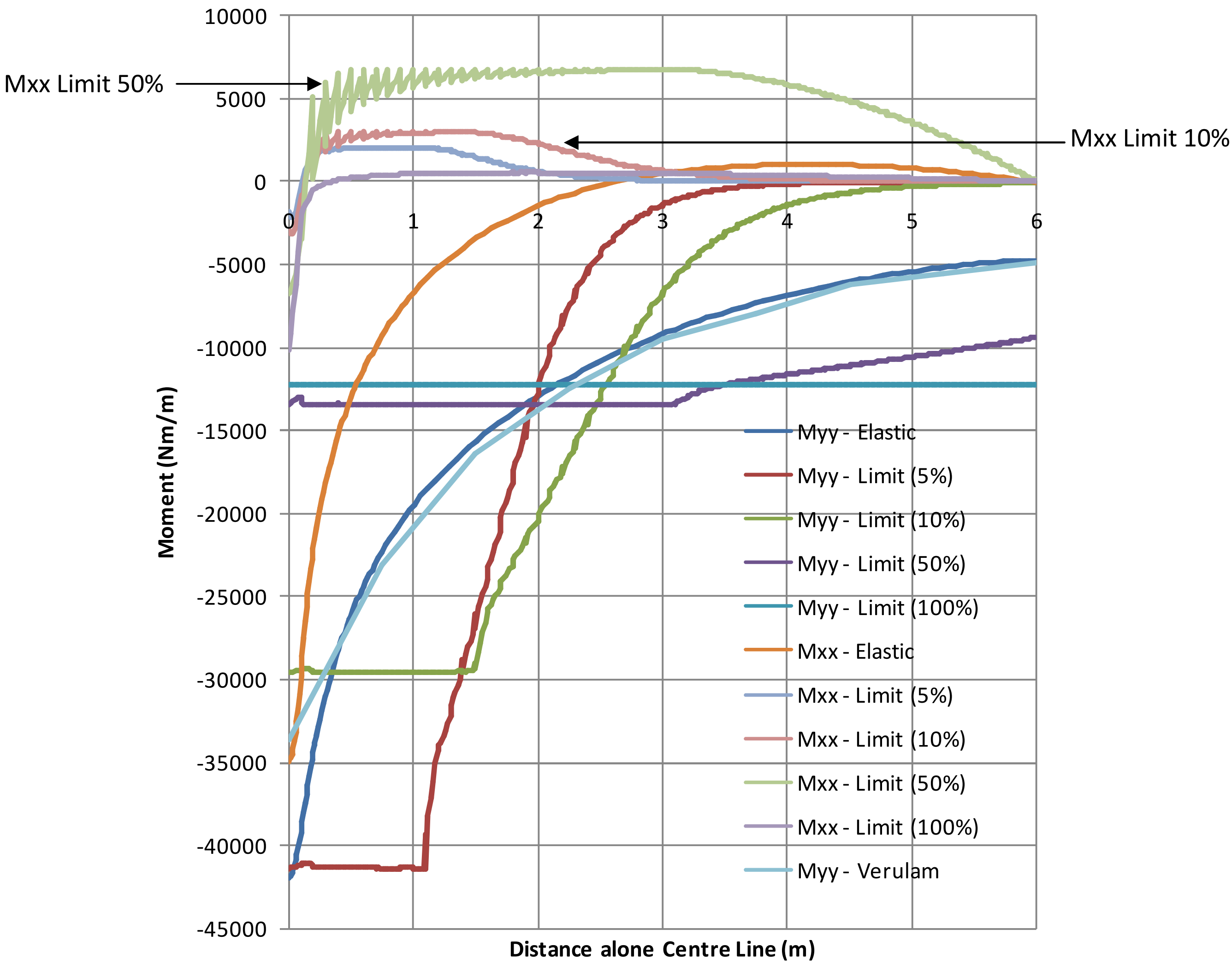



**Figure 8:** Distributions of Mxx and Myy along centre line (Elastic and Plastic)

## Closure

We have tried to show in the original letter and now in this supplement that the application of equilibrium finite element methods (elastic and/or plastic), provide rational and safe answers to many of the questions faced by practicing structural engineers.

It is clear from this exercise that there are considerable differences between finite element results, which we believe to be close to theoretical elastic solution, and methods based on finite differences or grillage models. Finite element software is widely available and should now be an everyday tool for the practicing structural engineer.

Finite element techniques can be extended to plastic methods which, when based on equilibrium, seek lower-bound solutions. This enables the engineer to explore the potential benefits of moment redistribution. Such methods provide a rational and safe approach to answering questions such as that posed in the original Verulam letter regarding the effective width of slabs.