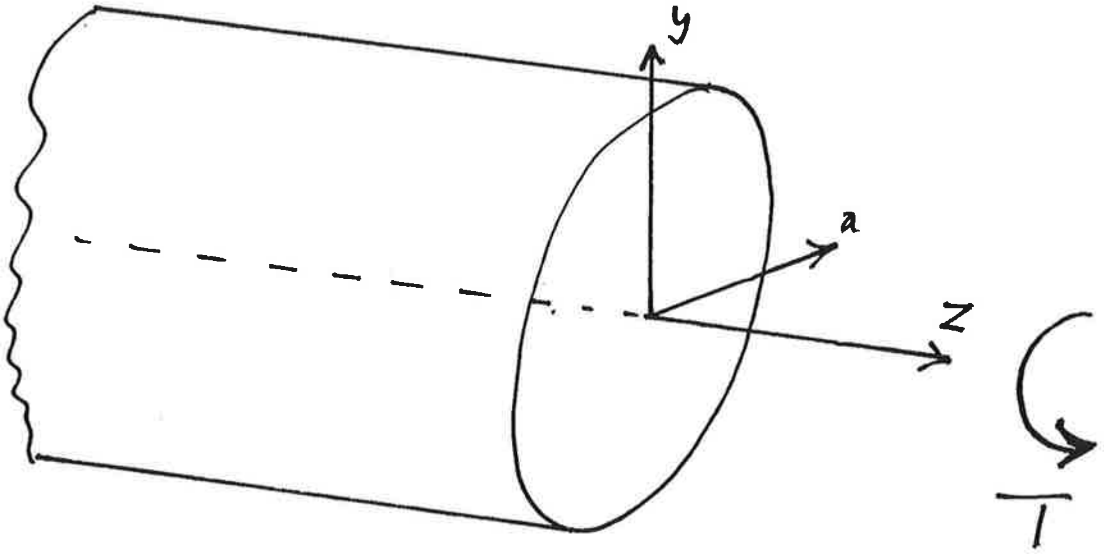
# 8. Torsion

Consider a prismatic member, of arbitrary but solid cross-section, loaded by torsion about its axis. i.e – no holes

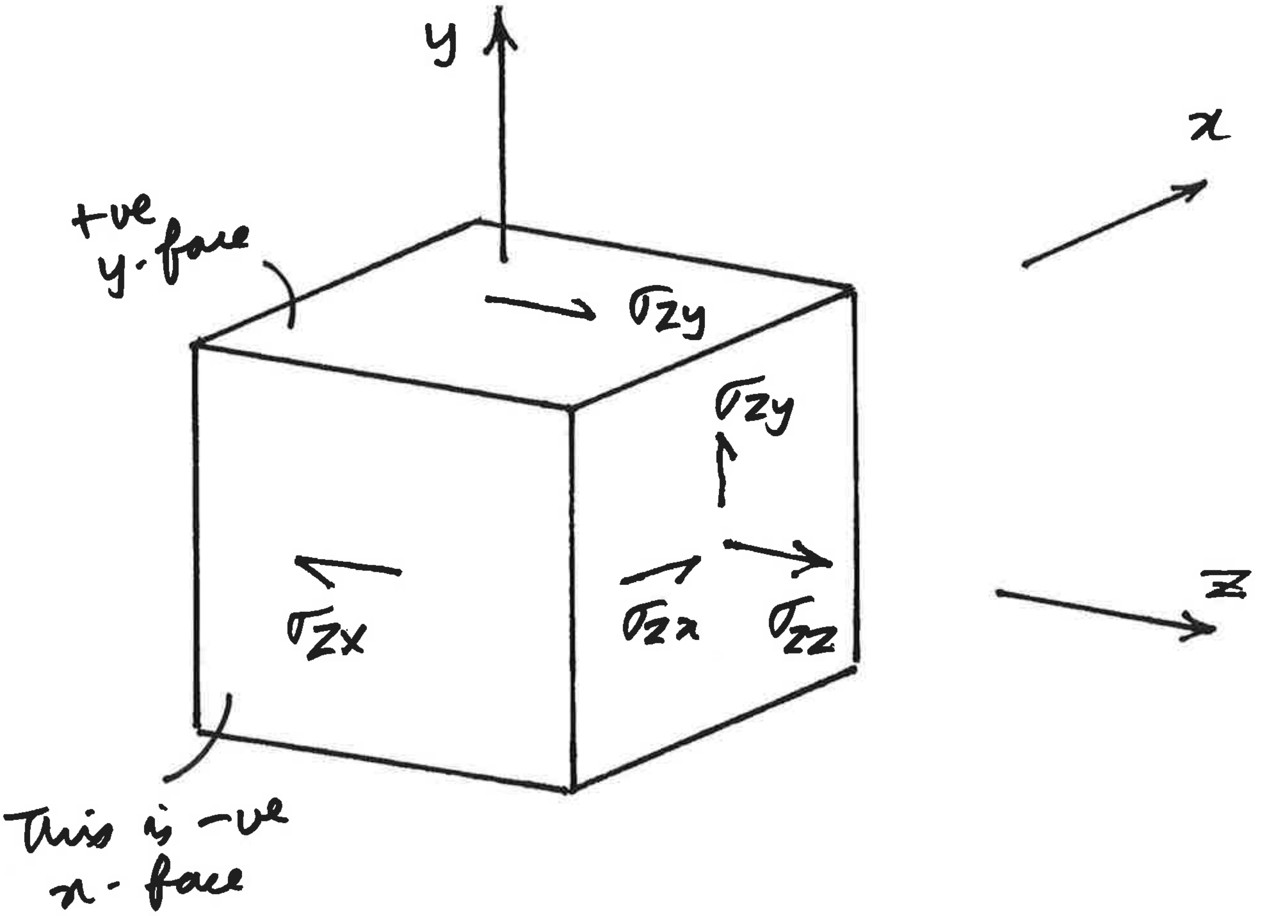
(We will not discuss here where that axis is – that will be covered in 3D4 where it will be shown that the location is important and specific.)



If the torque is constant, every cross-section must be subjected to the same torque.

All the stresses must be carried across each cross-section.

Consider a small element in that cross-section.

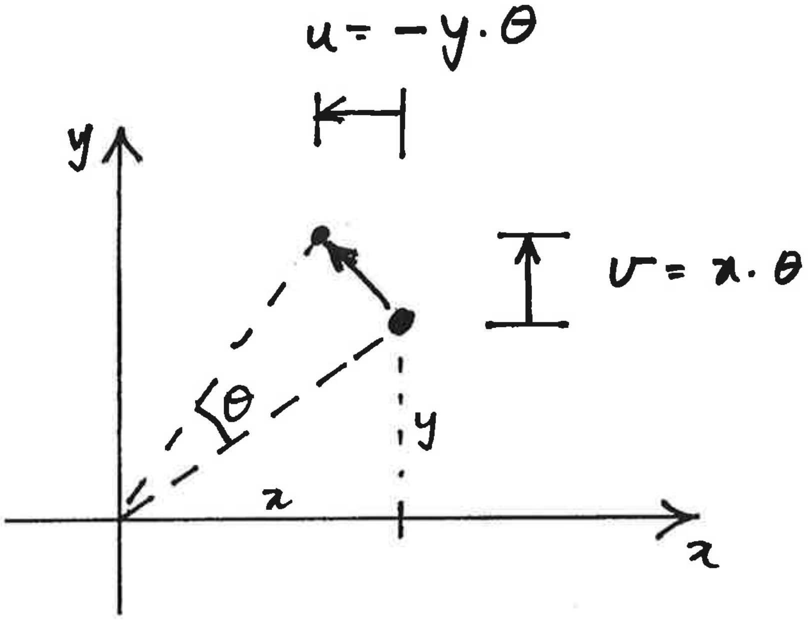


Equilibrium in the z-direction gives

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0$$

## Displacements

If the section rotates by an angle $\theta$ without distortion,
which is valid for thick sections, but not always for thin ones,
then the displacements in the $x$-$y$ plane are easy to find.



But the section will also warp by an (as yet) unknown amount. Different parts of the cross-section will displace axially by different amounts.

Note that this breaks the assumption about "plane sections remaining plane", which is valid for bending but not for torsion.

The warping displacement $w$ will be assumed to be proportional to the rate of change of the angle of twist:-

warping $$w = \theta'.f(x,y)$$ warping function (not yet known)

If the bar is subject to uniform torsion, $f(x,y)$ will be the same for all cross-sections.

Note that this is a very strong assumption. There are very important problems for which it is not true, and methods to deal with the exceptions will be covered in 3D4.

**Strains**

With these displacements and using the strain-displacement relations from section 3.

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \gamma_{xy} = 0$$

$$\varepsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \theta' \left( \frac{\partial f}{\partial x} - y \right)$$

$$\varepsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \theta' \left( \frac{\partial f}{\partial y} + x \right)$$

**Stresses**

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0$$

(so no normal stresses between fibres and no axial stresses)

$$\left. \begin{aligned} \sigma_{xz} &= G\theta'\left( \frac{\partial f}{\partial x} - y \right) \\ \sigma_{yz} &= G\theta'\left( \frac{\partial f}{\partial y} + x \right) \end{aligned} \right\}$$

N.B these are <u>not</u> a complementary pair of shear stresses.

Equilibrium equation becomes:-

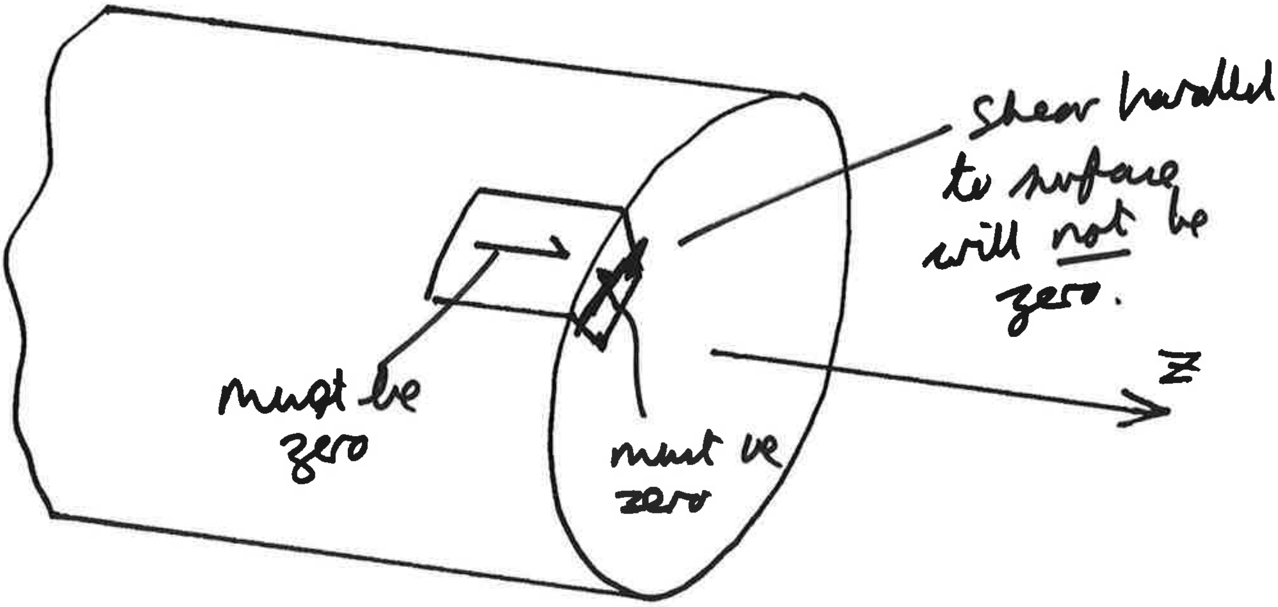
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0$$

or

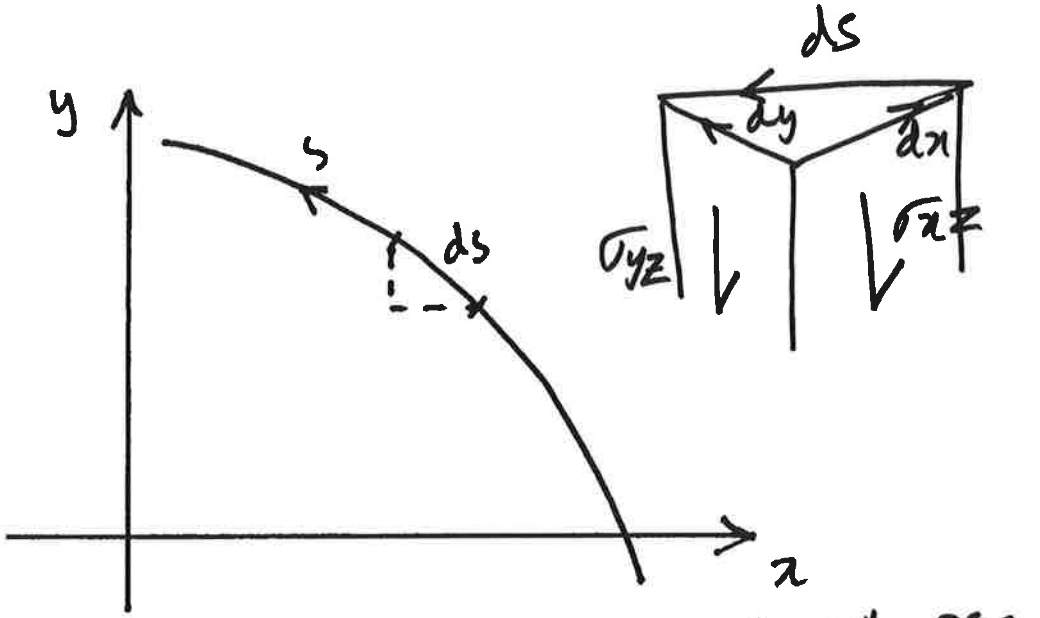
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

## Boundary Conditions

Shear stress must be parallel to the edge



Consider shear stresses on small element



Resolve stresses on small element to get zero shear on outer surface

$$\Rightarrow \qquad \sigma_{xz}\frac{dy}{ds}-\sigma_{yz}\frac{dx}{ds}=0$$

$$\left(\frac{\partial f}{\partial x}-y\right)\frac{dy}{ds}-\left(\frac{\partial f}{\partial y}+x\right)\frac{dx}{ds}=0$$

Can be turned into a boundary condition based on $\partial f/\partial n$ normal to the surface but complicated.

## Prandtl Stress Function

Prandtl (1903) suggested the use of a stress function $\psi$ which has the properties that:-

$$\sigma_{xz} = \frac{\partial \psi}{\partial y} = G\theta'\left(\frac{\partial f}{\partial x} - y\right)$$

$$\sigma_{yz} = -\frac{\partial \psi}{\partial x} = -G\theta'\left(\frac{\partial f}{\partial y} + x\right)$$

N.B.: $\psi$ defined by derivatives, not by absolute value

which means that the differential equation becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2G\theta'$$

and the boundary condition simplifies to

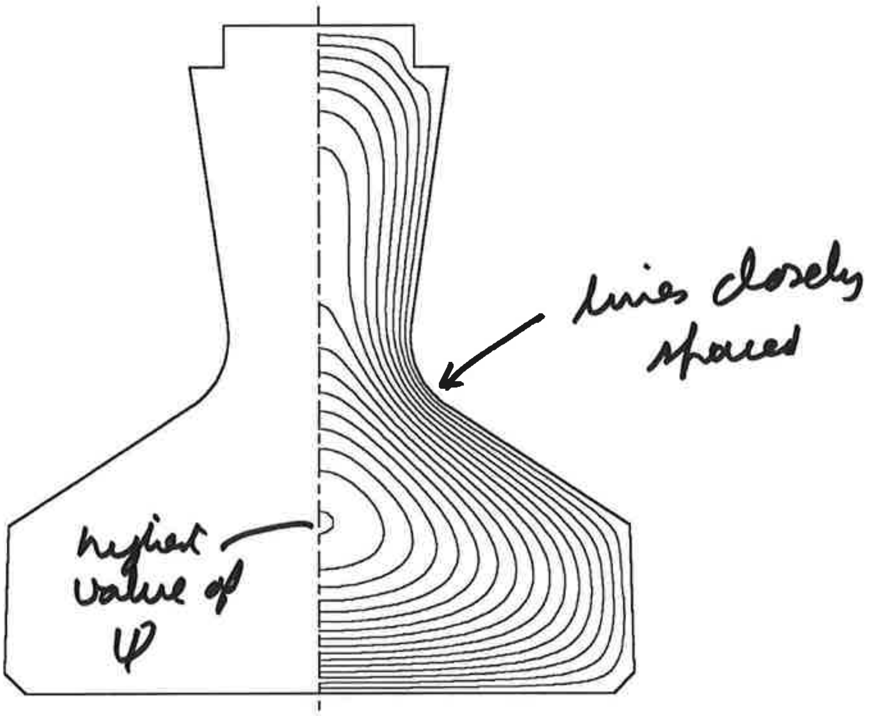
$$\frac{d\psi}{ds} = 0$$

i.e. $\psi$ is constant around the boundary. Normal to arbitrarily take $\psi = 0$ around an external boundary.

This equation is much easier to solve using finite difference or finite element programs.

Example ~~8~~115

Precast concrete beam – equal contours of $\psi$



Notice concentration of contours near re-entrant corner.

What does this mean? From definition of $\psi$

$$\sigma_{xz} = \frac{\partial \psi}{\partial y}$$

Magnitude of shear stress is proportional to slope of $\psi$ function.

So shear stresses are highest at edge near re-entrant corners and lowest at external corners and at the points furthest from an edge.

Direction of shear stress is parallel to contours.

No stress passes across contour lines. Could consider section as a set of nested tubes whose thickness varies.

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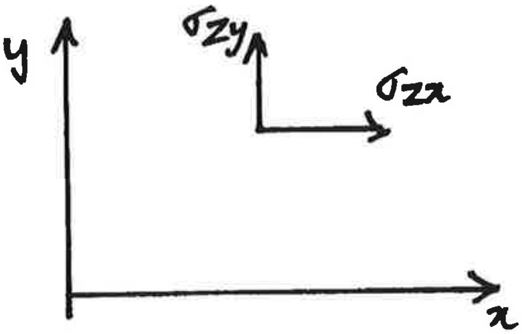
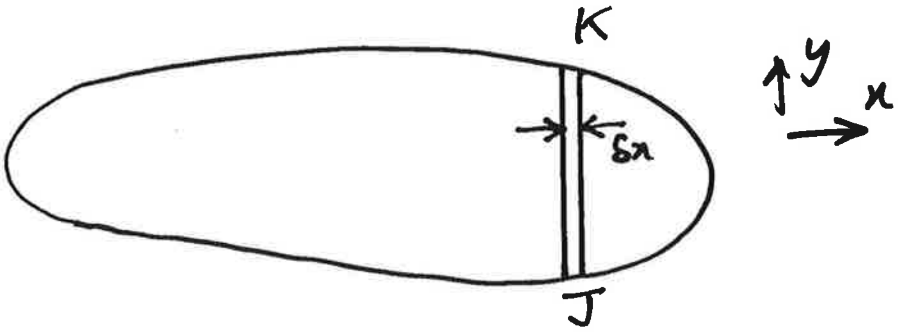
## Torque

What is the torque on the section?

Do the stresses produce a pure torque ?



Consider $\sigma_{xz}$ stresses in vertical strip between points J and K



Shear force in *x*-direction is

$$X = \iint \frac{\overset{\sigma_{xz}\downarrow}{\partial \psi}}{\partial y} dxdy = \int \left( \int \frac{\partial \psi}{\partial y} dy \right) dx$$

$$= \int (\psi_K - \psi_J) dx$$

But $\psi = 0$ on boundary

So, $X = 0$ which means shear force from this strip in x-direction is zero, so shear force from all strips must be zero.

Similar argument in other direction

So what is the torque about the origin? ↺

$$T_x = -\int\int y\frac{\partial\psi}{\partial y}dxdy = -\int\left(\int_J^K y\frac{\partial\psi}{\partial y}dy\right)dx$$

$$= -\int\left(\underbrace{\cancel{[y\psi]_J^K}}_{0} - \int_J^K \psi dy\right)dx = \int\psi dA$$

Similarly for $T_y$

So the total torque on the section is simply twice the volume under the surface defined by $\psi$

$$T = 2\int\psi dA$$

## Torsional stiffness

Torque $\qquad T = GJ\theta' = 2\int\psi dA$

So $\qquad J = \dfrac{2\int\psi dA}{G\theta'}$

$J$ is known as St. Venant torsion constant and is not the Polar 2nd Moment of Area.

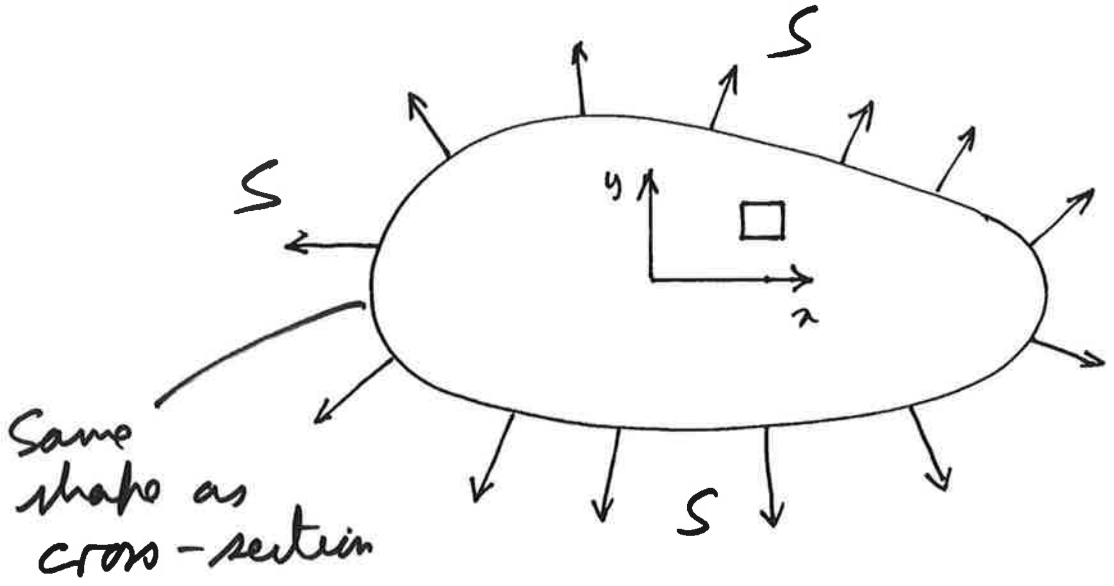
Normal procedure – set $G\theta' = 1$; set up and solve Poisson equation to get volume under surface (to get stiffness) and find the steepest slope to get maximum shear stress.

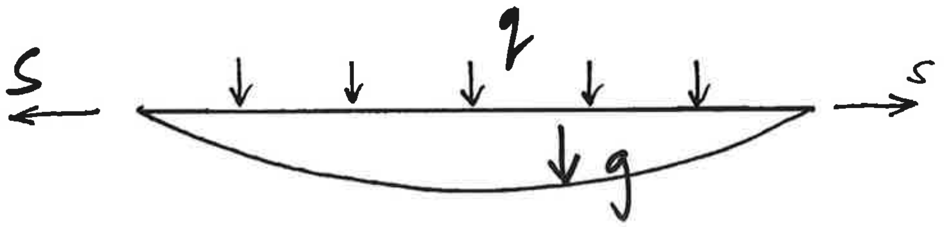
## Membrane analogy

(also called soap film analogy)

Useful for visualizing shape of function

Imagine a rubber sheet, stretched across a planar wire frame, in such a way that it has a uniform outward stress S at the edges

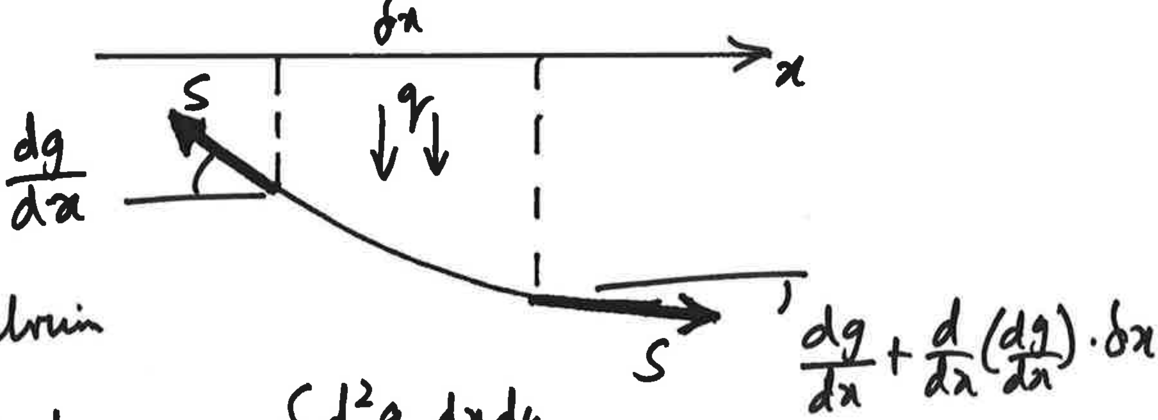




Now imagine that it is loaded by a uniform lateral pressure q

What shape does the sheet take up? Assume small deflections

Consider a small element in x-direction only



Equilibrium

$$q\cdot\delta x\,dy = -S\frac{d^2g}{dx^2}\cdot dx\,dy$$

( Similarly for term from y-direction

or $\dfrac{\partial^2 g}{\partial x^2}+\dfrac{\partial^2 g}{\partial y^2}=-\dfrac{q}{S}$ with $g=0$ on boundary

So the deflection of the rubber sheet ($g$) takes the same form as the Prandtl stress function $\psi$.

(Real) $2G\theta' \equiv {}^{q}\!/_{S}$ (Membrane analogy)

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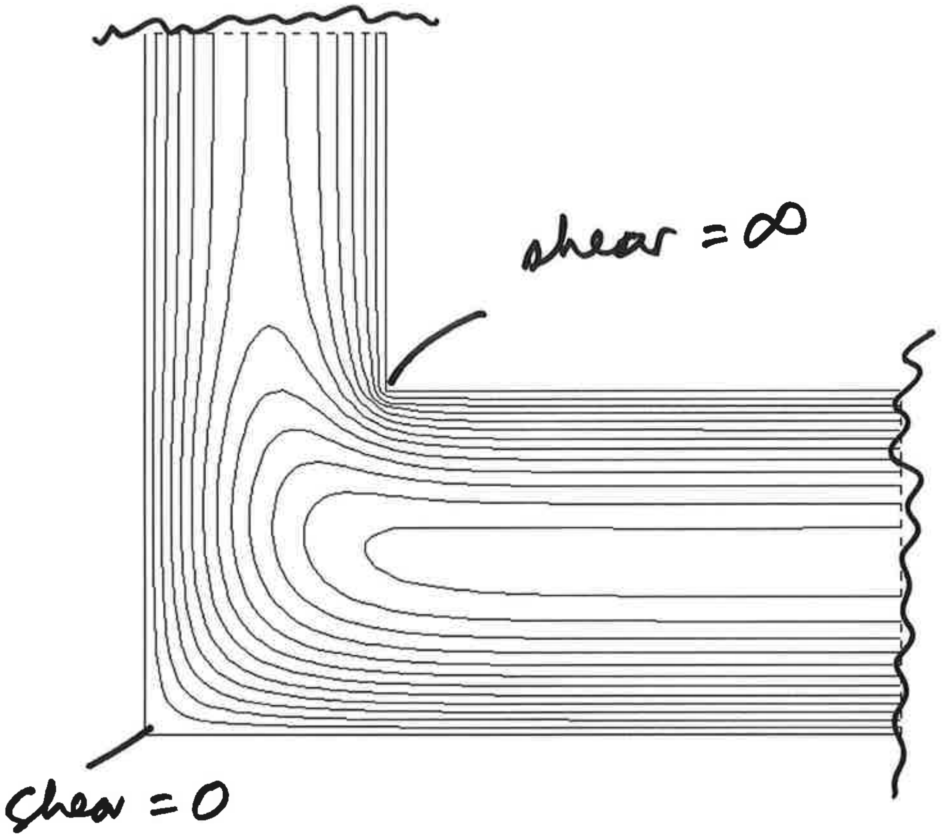
Easy to show then that shear stress at external corners is zero and at internal corners must be infinite

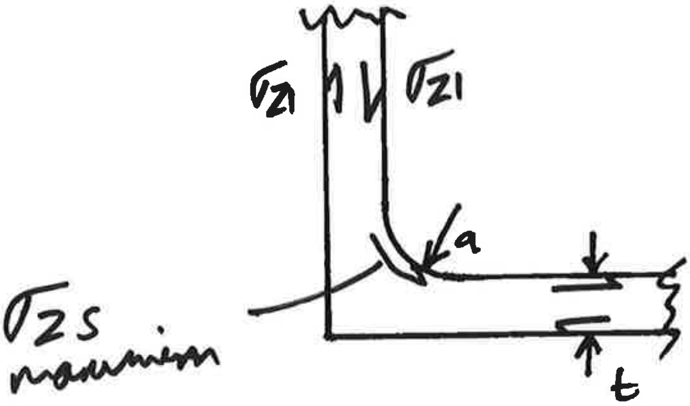


Note contours parallel to edges in long straight sections.

Be very careful of this effect – finite element analyses will not pick this up, or will give answers that depend on the size of the elements used.

Result is a function of the analysis method used and is not an accurate representation of what happens.

Rounded corners



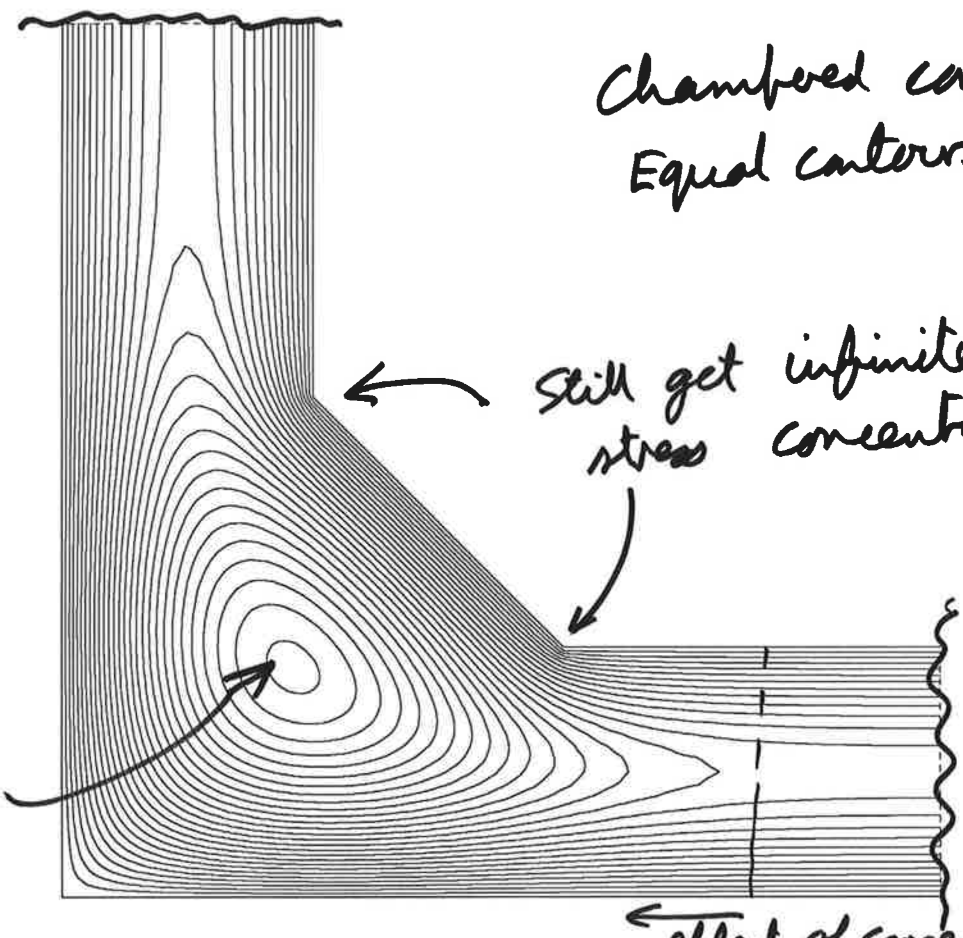
If corner is rounded the stress goes up but not as much

$$\sigma_{zs(\max)} = \sigma_{z1}\left(1 + {}^{t}\!/_{4a}\right)$$

(see Timoshenko & Goodier for analysis)

In parallel-sided elements, variation in $\psi$ becomes one-dimensional and varies parabolically across the section.

Very useful when analyzing sections made up from flat plates



Chamfered corner
Equal contours

Still get infinite stress concentrations

J ∝ volume

most of volume in this area

effect of corner

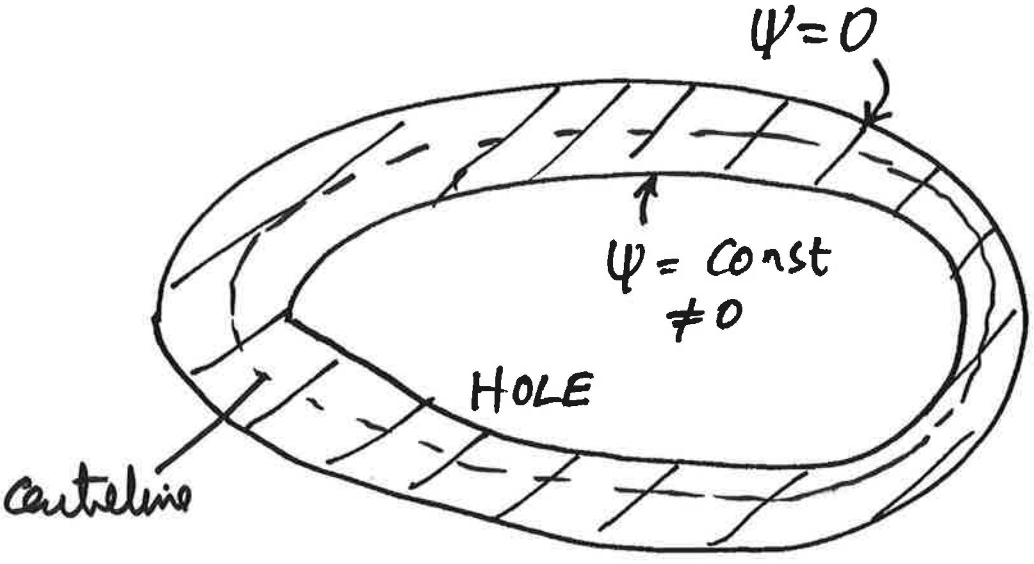
Variations concentrated in local areas at junctions, which can have major effect if section is thicker at that position.

Example of St Venant's principle.

effect of local variation in section is locally to change the stresse.

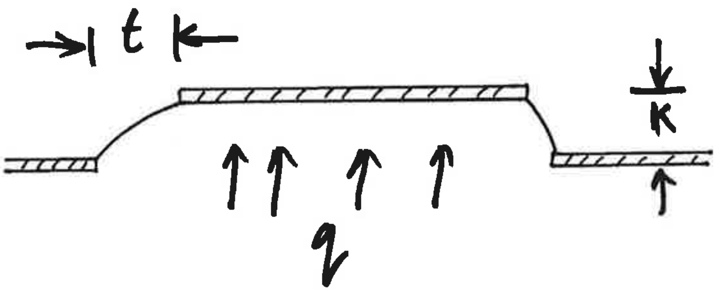
## Sections with holes

$\psi$ = constant on each boundary but will not be the same on inside and outside.

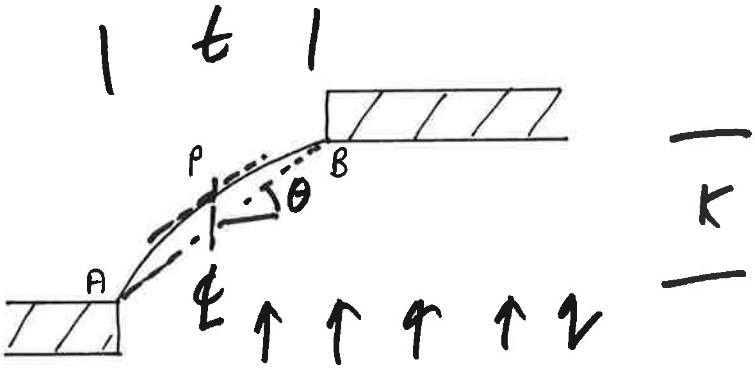


Can still show that $T = 2\iint\limits_{A}\psi.dxdy$ where integral is taken over whole area, including the hole. (same logic as before)

Membrane analogy. Imagine hole as a rigid plate that must remain parallel to the rigid plate around the outside. Assume that the tube is thin.



Consider area within centreline of the tube ($A_{cl}$)



For $t$ small APB is a parabola

so slope at P = slope of AB

Normal reaction/unit length of centre line of membrane

$$S\sin\theta \approx S\tan\theta = S\frac{k}{t} \qquad \text{since } \theta \text{ small}$$

So equilibrium of membrane gives

$$qA_{cl} = \oint S\frac{k}{t}dl = Sk\oint\frac{dl}{t}$$

because S and k are constants

Real

membrane analogy

Remembering $2G\theta' \equiv q/S$, then $J = \dfrac{2\int\psi dA}{G\theta'} = \dfrac{4S\int g dA}{q}$

Volume under membrane $\int g dA = kA_{cl}$

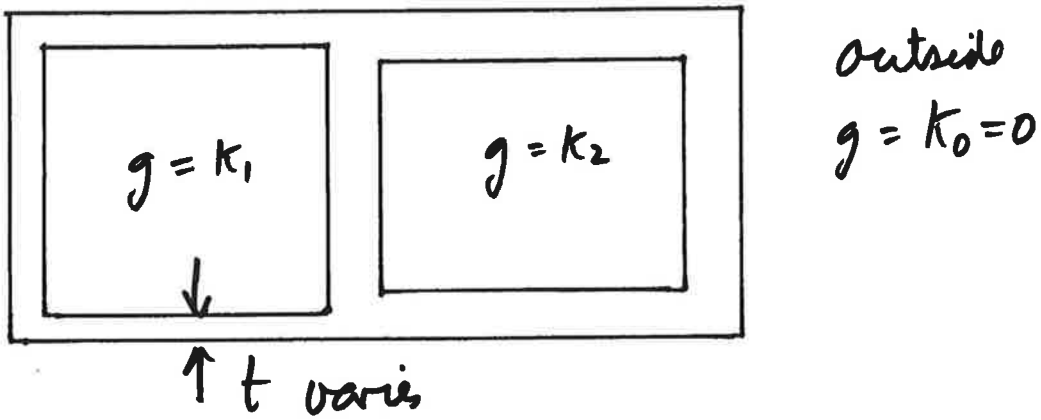
Leads to $J = \dfrac{4A_{cl}^2}{\oint\dfrac{dl}{t}}$ which is the formula in the data book for torsional stiffness of a thin tube

Note that this result has been achieved without consideration of displacements, so system is statically determinate. This is because there is only one rotation to be considered.

## Multi-cell tubes

These systems are now statically indeterminate. We must ensure that the rotation we impose on each cell is the same.

Apply membrane analogy as before. In each hole the displacement of the rigid plate is given by $g = k_i$ (and the outside can be considered a special case where $g = k_0 = 0$.



So in each piece of wall the normal reaction, per unit length, in the membrane is given by

$$\frac{S(k_i - k_j)}{t}$$

constant in each wall segment

can vary over the section

where *i* and *j* are the areas on each side of the wall and *t* is the wall thickness (which can vary).

Equilibrium of each plate gives

$$qA_i = S\oint \frac{(k_i - k_j)}{t} dl$$

Rate of twist (which must be the same for all tubes)

$$G\theta' = \frac{q}{2S} = \frac{1}{2A_i}\oint\frac{(k_i - k_j)}{t}dl$$

where integration is taken round wall of tube $i$.

If $N$ cells, there are $N$ equations like this which allow the values of $k_i$ to be found.

Solve for $k_i$ & then proceed as for single tube.