NAFEMS Benchmarks for Plastic Collapse of Square Metal Plates

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Abstract

Benchmark problems with either known theoretical solutions or, alternatively, numerical solutions of sound provenance are of importance to the practising engineer. They are used to verify the finite element system that the engineer is using and also enable him/her to determine the level of mesh refinement required to capture the structural response to the level of accuracy required. NAFEMS offers a range of benchmark problems in their publications together with finite element solutions to the problems. In checking NAFEMS benchmark problems for the plastic collapse of metallic plates, the author found that not only were the reference solutions incorrect but also the finite element solutions provided were insufficiently converged. The reasons behind these issues are explored in this article and provide an interesting illustration of the level of care required when preparing and publishing benchmark problems. The author is a member of the Education & Training Working Group at NAFEMS and has informed the organisation of the errors in their benchmark. NAFEMS will reference this publication in an updated version of the publication containing this benchmark.

Introduction

NAFEMS offers engineers a set of selected benchmarks for material non-linearity, [1], which includes the plastic collapse of plates obeying the von Mises yield criterion. The benchmarks involve a uniformly loaded square plate either simply supported (NL7A) or clamped around all edges (NL7B) – see Figure 1. Neither problem has a known theoretical solution, although this can be bounded rather closely, and so, in [1], *reference* solutions are provided from the literature which are then compared with *target* solutions obtained by finite element (FE) analysis.

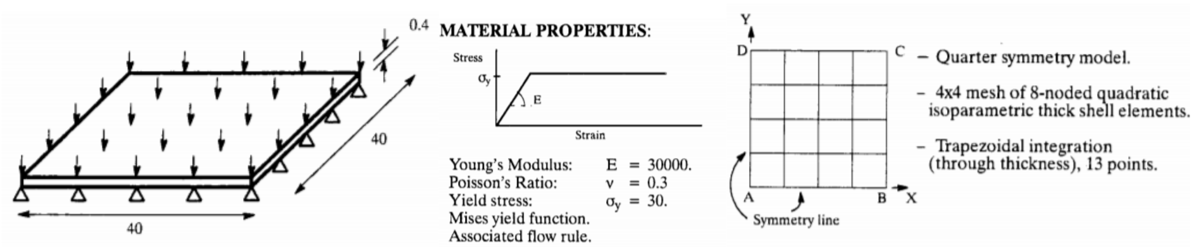


Figure 1: Geometry, material and mesh for NL7 Benchmarks from [1]

In 1997, software developed for the geometric optimisation of yield line patterns, [2], was used to check the reference solutions and the collapse load for NL7B was discovered to be about 12% too large, [3]. At that time and without access to software capable of approximating the collapse load for the von Mises criterion the result for NL7B was left unchallenged. Since then Ramsay Maunder Associates (RMA) has developed software based on an equilibrium finite element (EFE) formulation for the lower bound limit analysis of reinforced concrete slabs which, when homogeneously and isotopically reinforced, are governed by the maximum principal moment or square yield criterion, [4]. This was subsequently extended to include the von Mises or elliptical yield criterion which is more appropriate for metallic plates. RMA:EFE, has been thoroughly verified for both yield criteria through, *inter alia*, blind experiment reported in [5]. The benchmark problems have recently been reanalysed using this software tool and whilst there is good agreement for NL7A, the target solution offered for NL7B is about 17% too large.

There are a number of issues with the NAFEMS benchmarks and these will be scrutinised in this article. The reference solution is obtained through crude yield line analysis leading to a significant upper bound prediction of the theoretical collapse load. The reference solution is also based on the square yield criterion whereas the target solution is based on the elliptical criterion. The differences in these two yield criteria is such that there is no reason to believe that the theoretical collapse loads obtained using the two criteria will be the same or that one will always be more conservative than the other. The target solution is obtained using FE analysis on a very coarse mesh and solution verification presented in this article shows the target solution to be significantly in error when compared with the bounds on the theoretical solution presented in this article.

In addition to providing suitably verified target solutions and updated reference solutions for these benchmarks, this article aims to expose some of the reasons why the errors, particularly for NL7B exist and why they may not have been picked up since they were first published in 1993. These benchmark problems are of particular interest to structural engineers who are often tasked with the design and/or assessment of steel plates and might use them as software verification problems. The literature on plates already includes a number of errors and approximate solutions that are likely to mislead the practising engineer – see [10] for example. NAFEMS regularly review their benchmarks and such a review is currently in progress. It is hoped that the findings of this article will help NAFEMS in this process and lead to a sound set of results which the practising structural engineer can trust.

Limit Analysis Techniques

Limit analysis is a method of analysis through which the engineer can get directly to the plastic collapse load for a structural member. It adopts a rigid, perfectly-plastic material model and, computationally, it is extremely efficient but, as with any other forms of numerical simulation, the solutions are only approximations to the theoretically exact collapse loads.

The yield line technique (possibly the first limit analysis approach) dates from the first part of the last century as a hand calculation method for predicting the collapse load for reinforced concrete slabs. The engineer must postulate a (kinematically admissible) collapse mechanism, in the form of a yield line pattern, from which the corresponding collapse load can then be determined. In terms of the plasticity theorems, the method is an upper bound technique so that unless the postulated collapse mechanism corresponds to an exact mechanism (the collapse mechanism generally not being unique – see [13] for example), the predicted collapse load will be greater than the theoretically exact value. This is one of the issues with the reference solution for NL7B – even if the square yield criterion were appropriate for steel, which it is not, the postulated collapse mechanism is not a good representation of the actual mechanism and, thus, the predicted collapse load is significantly upper bound.

The software tool RMA:EFE, on the other hand, works differently. The solution process begins with an *equilibrating* particular solution (a moment field in equilibrium with the applied load) and then adds self-balancing (hyperstatic) moment fields, the amplitudes of which are chosen so as to maximise the load carrying capacity of the slab without violating the appropriate yield criterion. The moment field at collapse is statically admissible and so it is, therefore, a lower bound technique which will always, irrespective of mesh refinement, offer the engineer a safe prediction of the collapse load. The predicted collapse loads from the two techniques bound the theoretically exact value for a given slab as was demonstrated in [6].

Yield Criteria

Materials behave differently when strained beyond the elastic limit and obey, at least approximately, different yield criteria. Reinforced concrete (RC), when under-reinforced, is a ductile composite material that obeys closely the Nielsen Bi-Conic yield criterion. When viewed in principal moment space and assuming homogeneous and isotropic reinforcement, the yield curve is identical to that of the maximum principal moment criterion, i.e., it is a square. Ductile metals, on the other hand, more accurately obey the von Mises criterion this being elliptical in principal moment space.

The concept of an equivalent moment is useful in explaining the different yield criteria. Given the cartesian moments at a point in a plate, m_x, m_y and m_{xy}, the principal moments, m_1 and m_2 ($m_1 \geq m_2$) are simply determined. Different yield criteria lead to different definitions of the equivalent moment, m_E, and these are defined in terms of the principal moments in Eq. (1). In order to distinguish between equivalent moments based on different yield criteria, a superscript identifying the shape of the yield curve for the particular yield criterion adopted has been added – *S* for square and *E* for elliptical.

$$m_E^S = \max(|m_1|, |m_2|) \quad \text{Square} \qquad (1a)$$

$$m_E^E = \sqrt{m_1^2 - m_1 m_2 + m_2^2} \quad \text{Elliptical} \qquad (1b)$$

In a linear elastic analysis, the engineer will normally obtain a contour plot of equivalent moment from which the maximum value(s) can be obtained. The moment to cause first surface yield, m_e or to cause first section yield, m_p, is defined in Eq. (2) where S_y is the uniaxial yield stress for the material and t is the plate thickness which is assumed uniform over the plate.

$$m_e = \frac{S_y t^2}{6} \qquad (2a)$$

Moment capacity (elastic) for a point in a uniform thickness plate to limit the load to cause first surface yield

$$m_p = \frac{S_y t^2}{4} \qquad (2b)$$

Moment capacity (plastic) for a point in a uniform thickness plate to limit the load to cause first section yield

With these definitions of the equivalent moment and the yield moments for the surface and section of his plate, the engineer can, by scaling his/her linear elastic results, determine the elastic and section limit loads as shown in the first two rows of Table 1 where $\widehat{m}_E$ is the maximum value of the equivalent moment in the plate.

Load	Meaning	Relation	Description
q_e	First *surface* yield ($\widehat{m}_E = m_e$)		Elastic limit load
q_p	First *section* yield ($\widehat{m}_E = m_p$)	$q_p = 1.5 q_e$	Section Limit Load
q_c	*Member* collapse ($m_E = m_p$ along sufficient (yield) lines to create a collapse mechanism)	$q_c \geq q_p$	Member Limit Load

Table 1: Definition of limit loads for a uniform thickness plate

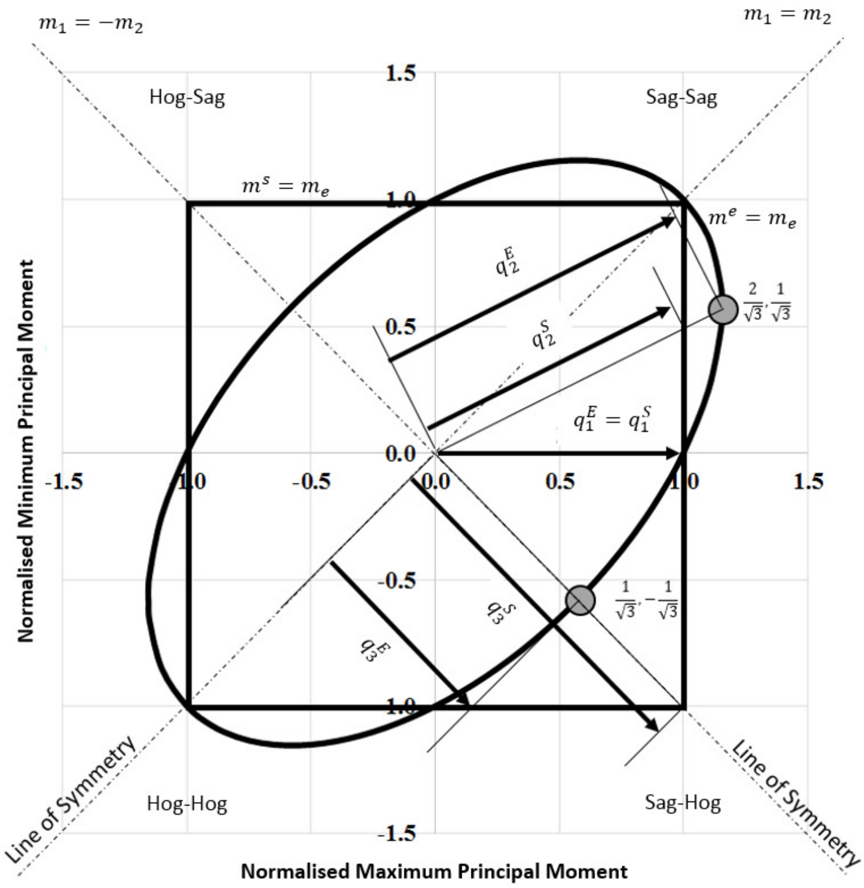
Based on a rigid, perfectly-plastic material model the member will collapse at a load q_c. The same collapse load will be achieved through an incremental FE analysis with an elastic, perfectly-plastic material model irrespective of any initial self-equilibrating residual stresses present in the plate due, say, to manufacturing processes. It is noted that in the presence of initial residual stresses the loads to cause first surface and section yield, when based on an analysis assuming an initially unstressed member, are really only notional figures.

The plastic collapse load, as defined above, is likely to possess considerable conservatism since the strengthening phenomena of material strain-hardening and plate member membrane-action are not taken into account. Studies conducted by the author on the clamped plate configuration indicate that when membrane-action is included the actual collapse load is at least twice that predicted by a limit analysis which excludes membrane-action.

The yield curves plotted in Figure 2 show that the curves cross each other. This means, depending on the particular principal moment state at the point of maximum moment, the elastic limit load is dependent on the yield criterion selected and that one criterion is not a conservative form of the other.

Taking the two extreme cases shown in Figure 2, (q_2 and q_3) the range of the ratio of loads to cause first surface yield for the two different yield criteria is as shown in Eq. (3).

$$\frac{1}{\sqrt{3}} \leq \frac{q_e^E}{q_e^S} \leq \frac{2}{\sqrt{3}} \qquad\qquad 0.577 \leq \frac{q_e^E}{q_e^S} \leq 1.155 \tag{3}$$



The principal moments are normalised by dividing by the appropriate moment capacity; m_e if one is considering first surface yield or m_p if first section yield is to be considered.

Figure 2: Yield diagram for a uniform thickness cross section

There has been a tendency, particularly in the older literature on elastic analysis of plates, to tabulate only the maximum principal moment. This forces the engineer to adopt the square yield criterion in predicting the load to cause first yield and this could be significantly different from that achieved using the more appropriate elliptical yield criterion. Taken to the rational conclusion, with only one principal moment available, the engineer cannot know where in the range of Eq. (3) his/her solution lies. It could be at either end whereas in reality the truth could be at the other end of the range. This implies, in the absence of any further information, that significant uncertainty exists in his elastic limit load. The percentage error in adopting q_e^S as the elastic limit load compared to q_e^E is defined in Eq. (4).

$$e = \frac{q_e^E - q_e^S}{q_e^E} \cdot 100\% \tag{4}$$

The extreme values of the error are presented in Table 2.

	Scenario Number 1	Scenario Number 2
q_e^E	$\frac{2}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
q_e^S	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$
e	**+50%**	**-100%**
	Safe but Uneconomic	**Unsafe**

Table 2: Extreme values of the error in elastic limit load

The percentage errors presented in Table 2 are significant. The true elastic limit load, q_e^E, may be under-predicted by 50% or over-predicted by 100% if q_e^S is adopted as a surrogate load! In an allowable stress design (ASD) philosophy, these potential errors have significant implications on, respectively, the economy of the plate (selection of plate thickness in a design) and the safety of a plate in assessment. Of course, in a limit state design (LSD) philosophy, the ultimate limit state (ULS) condition would not generally be the elastic limit but, rather, the plastic limit load.

The error analysis shown above is for the elastic limit load. However, similar conclusions can be drawn for the plastic limit load such that values derived from a limit analysis using the square yield criterion would have the same level of uncertainty, i.e., -100% -> +50%, if used as a surrogate for the load derived with the elliptical criterion. The author has examples of plate configurations which lie at both ends of the range!

Collapse Loads for Square Yield Criterion

There are known theoretically exact solutions for the two plate configurations considered when the square yield criterion is used. These are presented in Eq. (5) where q_c is the collapse load for a plate of side length L and yield moment m_p.

$$q_c = \frac{24m_p}{L^2}$$ Simply Supported Case (NL7A) - Square Yield Criterion (5a)

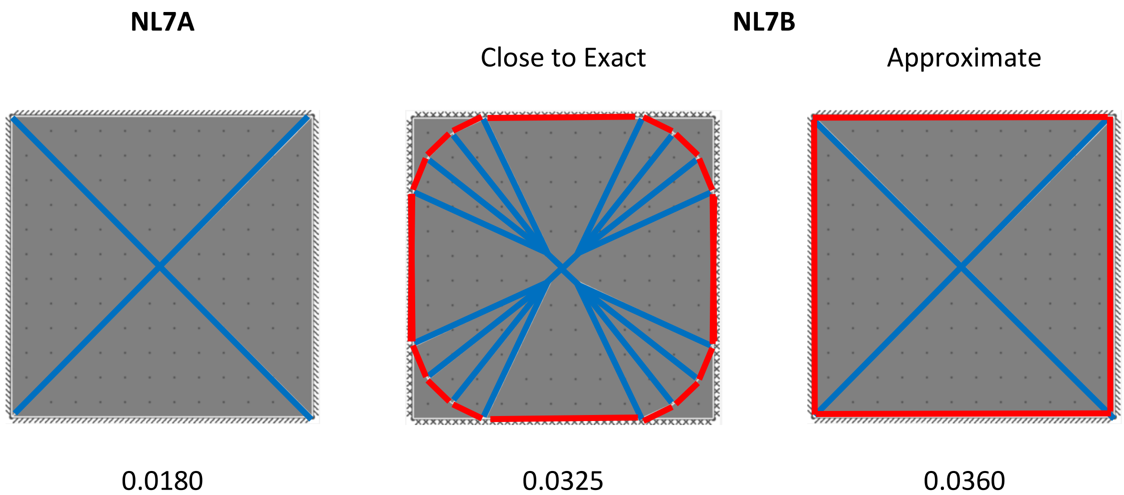
$$q_c = \frac{42.851m_p}{L^2}$$ Clamped Case (NL7B) - Square Yield Criterion (5b)

The reference solutions for the two benchmark problems are derived from yield line solutions. The yield line solutions for these plates were also presented in [3] where solutions were obtained using geometrical optimisation of the yield line pattern in order to find the least upper bound collapse load. These three solutions are presented in Table 3.

Plate Configuration	Reference from [1]	Yield Line from [3]	Theoretically Exact
NL7A	0.018	0.0180	0.01800
NL7B	0.036	0.0322	0.03214

Table 3: Collapse loads (Pa) using the Square yield criterion

The reference solution for NL7B has a collapse load that is about 12% greater than the theoretically exact value. The reason for this is that it comes from a yield line calculation based upon a poorly postulated collapse mechanism. The mechanism used is shown in the third column of Figure 3. The 'exact' collapse mechanism is of the type shown in the second column and involves fan type mechanisms in the corners, [7].



Sagging yield lines are shown in blue hogging yield lines in red.

Figure 3: Collapse mechanisms and loads for the plate configurations (Square Criterion)

Collapse Loads for Elliptical Yield Criterion

As noted previously, the use of different yield criteria will generally lead to different collapse loads albeit that the underlying yield line collapse mechanism is identical. Whereas theoretically exact solutions are known for these two plate configurations when using the square yield criterion, this is not the case for the elliptical yield criterion. As such, we will need to rely on numerical solutions of good provenance to closely bound the theoretical solution. For this purpose we will use the upper bound results presented in [8] and lower bound results generated by RMA:EFE. Bounded values for the collapse loads are given in Eq. (6).

$$25.01 \leq \frac{q_c L^2}{m_p} \leq 25.02$$ Simply Supported Case (NL7A) - Elliptical Yield Criterion (6a)

$$43.97 \leq \frac{q_c L^2}{m_p} \leq 45.29$$ Clamped Case (NL7B) - Elliptical Yield Criterion (6b)

The lower and upper bound collapse loads from Eq. (6) are compared with the target solutions in Table 4.

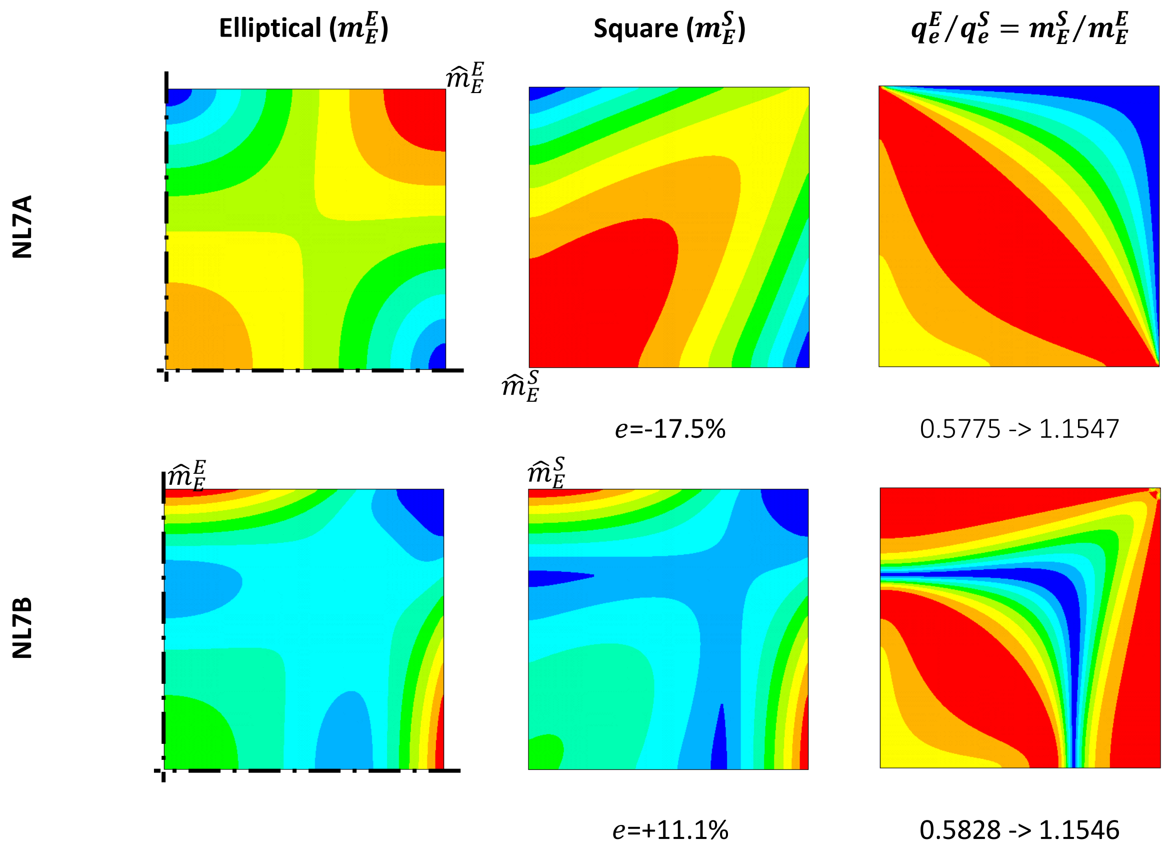
Plate Configuration	Target from [1]	Lower Bound	Upper Bound
NL7A	0.01877	0.01875	0.01877
NL7B	0.03852	0.03297	0.03397

Table 4: Collapse loads (Pa) using the Elliptical yield criterion

Thus, whilst by serendipity the target solution for NL7A lies within the bounds, the solution for NL7B is some 13% greater than the lowest upper bound value!

Collapse load by Incremental Elastic, Perfectly-Plastic Analysis

Before looking at the incremental analysis we will consider the results from a linear elastic analysis. Theoretical elastic solutions for NL7A and NL7B are given, in the form of infinite series, respectively by Navier and Henky, [9]. Note that whereas closed-form theoretical solutions are very useful for the practising engineer, those that are offered as infinite series are more problematic and in terms of the requirement to undertake convergence studies are not too different from a FE analysis, [14]. Contour plots of equivalent moments are shown in Figure 4. It is noted that for the simply supported configuration (NL7A) the location of the maximum equivalent moment is dependent on the selected yield criterion. For the clamped plate (NL7B) the location of the maximum equivalent moment is independent of the yield criterion and always lies at the centre of the sides.



The contours range from blue (minimum) to red (maximum).

Figure 4: Contours of elastic equivalent moments for different yield criteria

The greater the maximum value of the equivalent moment, the lower the predicted collapse load, i.e., the quantities are inversely proportional as indicated in Eq. (7).

$$q_e \propto \frac{1}{\hat{m}_E} \tag{7}$$

The third column of Figure 4 shows contour plots of the ratio of the equivalent moments based on the square and elliptical yield criterion. From the inverse proportionality of Eq. (7), this is the same as a plot of the ratio of the elastic limit loads based on the particular moments at a given point within the plate. The range of values of this ratio is shown in the figure and sits within the theoretical range given in Eq. (3). Interestingly, the bounds for the plate configurations considered here are close to the theoretical bounds and this demonstrates the importance of selecting the correct location in the plate and the correct yield criterion when determining the elastic limit load. The percentage error in using the square yield criterion as opposed to the elliptical criterion to predict the elastic limit load is also shown in the figure.

The majority of practising engineers do not have access to bespoke limit analysis software tools like RMA:EFE which adopt a *rigid*, perfectly-plastic material model. Nonetheless, they can make use of conventional FE systems and use an *elastic*, perfectly-plastic material model in an incremental analysis. The result achieved in terms of the collapse load should be identical although the approach requires sound verification both of the mesh and the incremental loading procedure and will take much longer in processing.

The plastic collapse loads, q_c, as predicted by various FE systems, are shown in Figure 5. Three commercial systems (CS) were used together with RMA:EFE.

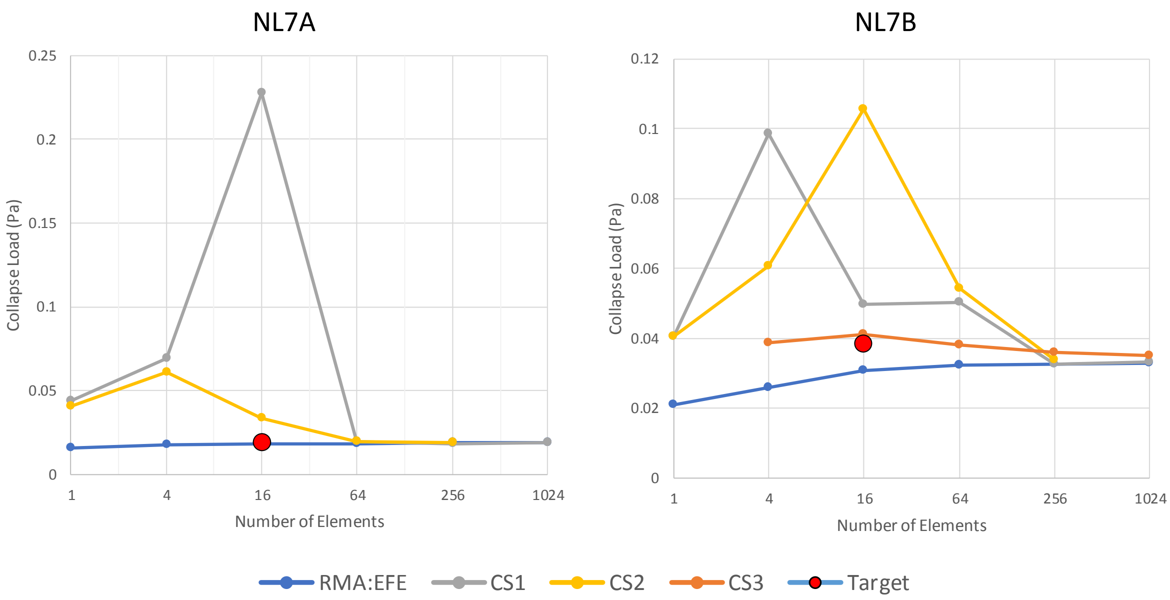


Figure 5: Convergence of collapse load for elliptical yield criterion

For both NL7A and NL7B, all FE solutions converge to more or less the same value which lies within the bounds shown in Table 4. RMA:EFE converges from the safe side of the solution, i.e., from below the true value whereas the three CSs converge from above, i.e., from the unsafe side. CS1 and CS2 seem to struggle with coarser meshes and exhibit a divergent pre-asymptotic behaviour prior to converging to the true solution. The number of elements required for CS1 and CS2 to produce reasonably accurate solutions is, respectively, 8x8=64 and 16x16=256 for NL7A and NL7B. Whereas the behaviour of CS1 and CS2 is rather similar, CS3 demonstrates a much smoother convergence with significantly better results for the coarser meshes. The result from CS3 for the 4x4=16 element mesh is very close to the NAFEMS target solution indicating that the formulation adopted in the software used in [1] might be rather similar to that in CS3.

Discussion

This article offers confirmation of the correctness, albeit by serendipity, of NAFEMS benchmark NL7A and a necessary correction for NL7B. It is interesting that it has taken the author over 20 years from the initial observation that NL7B might be incorrect to firmly demonstrating that this is in fact the case! A number of interesting points have arisen in this article which warrant further discussion.

The yield line technique, as an upper bound technique, requires the engineer to be extremely cautious in selecting an appropriate collapse mechanism or mode of collapse (yield line pattern) and then geometrically optimising this pattern. If this is not done properly then significant non-conservative over predictions of the collapse load are possible – see, for example, [6]. The yield line pattern assumed for NL7B ignored the complicated fan-type mechanisms around the corners of the plate and in doing so led to a 12% over prediction of the collapse load. At the time the benchmark was conceived (1990s) the exact solution for this problem had been in existence for almost 20 years, [7], and it is, therefore, surprising that this was not picked up by the authors or reviewers.

There is a widely propagated conceptual issue that assumes the yield line technique is equally as valid for metallic plates as it is for reinforced concrete slabs. Whilst the collapse mechanisms are generally identical for the two materials, the different yield criteria normally lead to different collapse loads. It is only a matter of serendipity that for the <u>square</u> plate configurations considered the collapse loads from the two yield criteria are so close. For the simply supported case the square and elliptical criteria gave collapse loads of 0.01800 and 0.01875 respectively which are within about 4% of each other. The simply supported <u>rectangular</u> plate considered in [5] showed a greater difference of about 7%. In both these cases the solution for the square yield criterion was less than that for the elliptical criterion and therefore might be considered a safe if not economical approximation. There are, of course, other plate configurations where it is unconservative to adopt the square criterion. It is thus of concern to the author that LSD codes of practice such as Eurocode 3, whilst explicitly allowing limit analysis, do not clearly state that the appropriate yield criterion must be adopted.

It is worth considering why the target solution offered in [1] for NL7B was so poor. The reason becomes obvious when shown the results from Figure 5; the results were simply unconverged and NAFEMS is guilty of publishing unverified results! Whilst the results do appear to converge, eventually, to the correct values, they converge in an oscillatory manner from above (far above for CS1 and CS2) the correct value. The results from RMA:EFE begin much closer to the correct solution and converge monotonically from below the correct solution. They are also achieved with significantly less computational effort – typically 1% accuracy can be achieved in 1 second of computer time. Thus, RMA:EFE is a software tool that satisfies two of the 'Big Issues' discussed at the NAFEMS World Congress in 2017. Barna Szabo's keynote speech defined these as:

Simulation Governance – What it takes for organisations to develop confidence in the results of their commercial simulation projects.

Democratisation – How to extend the benefits of numerical simulation to persons whose expertise is in other fields.

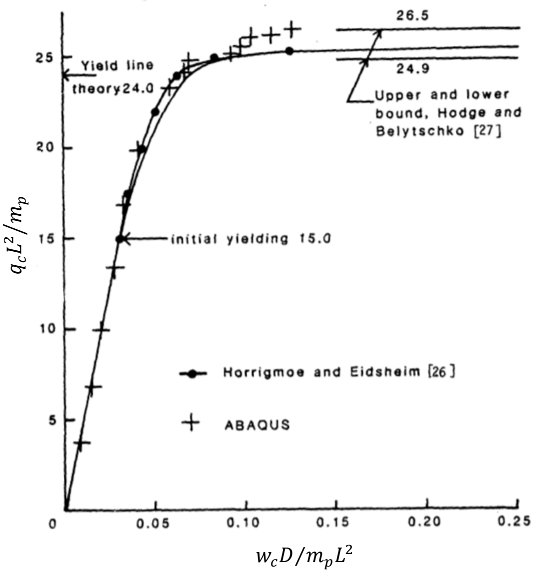
Practical Conclusions

The practising structural engineer involved in the design/assessment of steel plates should keep in mind the following points:

1) It is worth emphasising that, if the engineer must use first yield to define the ULS, then the differences possible by using the different yield criteria are potentially huge!
2) Since most plates will often contain significant residual moments due to manufacture and construction etc., a linear elastic FE analysis that does not include these will only lead to a notional elastic limit load.
3) The initial residual moments do not influence the plastic collapse load and in this sense, this is then a more reliable characteristic load than the elastic limit load.
4) The plastic collapse load, as determined through limit analysis or an incremental FE analysis, neglects the strengthening phenomena of strain-hardening and membrane-action and can normally be taken as a safe prediction of actual collapse.
5) Like the verification process undertaken in a proper FE analysis, published data also needs to be scrutinised and verified even if published by reputable bodies. Errors can creep into texts and even be propagated into future publications – see [10].
6) The appropriate yield criterion for steel plates is the elliptical one of von Mises. Caution should be exercised when using published results which are often based on the maximum principal moment or square yield criterion.
7) Most commercial FE systems adopt a conforming finite element (CFE) formulation which tends, for coarse meshes, to lead to significant over-predictions of the capacity of a plate. This means that mesh convergence studies are essential if the engineer is to produce safe designs and plate assessments.
8) Bespoke limit analysis software based on the lower bound principals of plasticity provide results that are guaranteed to be safe irrespective of mesh refinement. They thus conform with the ideas of simulation governance and democratisation. Mesh refinement with an EFE formulated limit analysis tool is still worthwhile since it leads to more accurate predictions of the collapse load and more economic plate designs.
9) Whilst overt conservatism has been a tradition in engineering in order to cope with the various uncertainties that exist, the drive to reduce CO_2 emissions will increasingly lead engineers to seek more economic design solutions, [11]. In order to do this, engineers will need reliable and accurate analysis software preferably of the type that ensures good simulation governance principles. The status quo of 'legacy' commercial CFE systems is far from this ideal, as illustrated in this article, and the door is thus open for disruptive technologies that do satisfy the more stringent simulation governance principles to enter the market place.

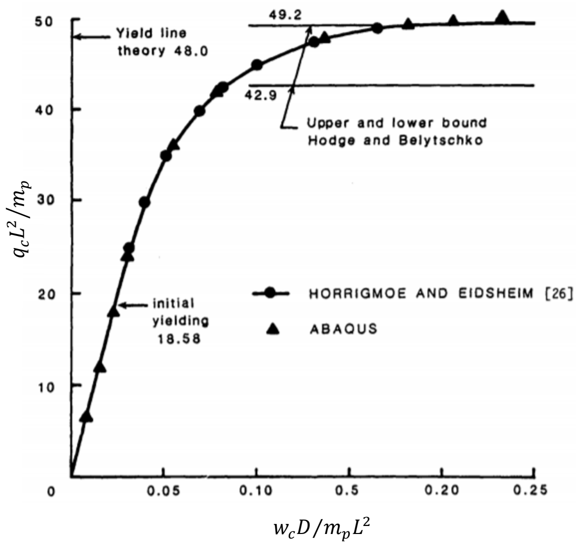
The reference solution used by NAFEMS for the plate configurations considered in this article came from [12] and the relevant load deflection plots from that reference are reproduced in Figure 6.

NL7A



$$25.01 \leq q_cL^2/m_p \leq 25.02$$

NL7B



$$43.97 \leq q_cL^2/m_p \leq 45.29$$

The plots show non-dimensional collapse load against non-dimensional deflection at the plate centre.

Figure 6: Reference solutions

Updated upper and lower bounds on the theoretical collapse load are shown below the plots in Figure 6 and are significantly tighter than those of Hodge and Belytschko shown in the figure. It is interesting to note that the ABAQUS results, for both NL7A and NL7B, which were generated by a 4x4=16 element quarter model mesh, appear to be converging to values above the upper bound.

Acknowledgement

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References

[1] NAFEMS, Selected Benchmarks for Material Nonlinearity Vol. 2, R0030e (1993).

[2] Ramsay, A.C.A. & Johnson, D., Geometric Optimisation of Yield-Line Patterns using a Direct Search Method, Structural & Multidisciplinary Optimisation, 14(2), 108-115, September 1997.

[3] Angus Ramsay, Automated Yield-Line Analysis for the Limit Analysis of Plates, NAFEMS Benchmark Magazine, July 1997.

https://www.ramsay-maunder.co.uk/downloads/Automated_Yield_Line_Analysis.pdf

[4] Edward Maunder & Angus Ramsay, Equilibrium Models for Lower Bound Limit Analyses of Reinforced Concrete Slabs, Computers & Structures, 108-109, 100-109, 2012.

[5] Angus Ramsay, NAFEMS Benchmark Challenge Number 2; Assessment of a Simply Supported Plate with Uniformly Distributed Load, NAFEMS Benchmark Magazine.

https://www.ramsay-maunder.co.uk/downloads/NBR_01.pdf

[6] Angus Ramsay, Edward Maunder & Matthew Gilbert, Yield Line Analysis of Reinforced Concrete Slabs: Is the 10% Rule Safe?, NAFEMS Benchmark Magazine, January 2015.

https://www.ramsay-maunder.co.uk/downloads/Yield-Line_Article_for_Benchmark_Magazine.pdf

[7] E.N. Fox, Limit Analysis for Plates; The Exact Solution for a Clamped Square Plate of Isotropic Homogeneous Material Obeying the Square Yield Criterion and Loaded by Uniform Pressure, Phil. Trans of Royal Society of London, Series A, Vol. 277, No. 1265, pp. 121-155 (August 1974).

https://royalsocietypublishing.org/doi/abs/10.1098/rsta.1974.0047

[8] Capsoni, A. & Corradi, L. 1999 Limit analysis of plates -a finite element formulation. Structural Engineering and Mechanics 8, 325–341.

[9] Timoshenko S.P. and Woinowsky-Krieger S. (1989) Theory of Plates and Shells (2nd ed.), New York, USA: McGraw-Hill.

[10] Angus Ramsay & Edward Maunder, An Error in Timoshenko's 'Theory of Plates & Shells', The Structural Engineer, June 2016.

https://www.ramsay-maunder.co.uk/downloads/Timoshenko-error.pdf

[11] John Orr, Minimising Energy in Construction: Survey of Structural Engineering Practice, Universities of Cambridge and Bath, 2018.

https://static1.squarespace.com/static/58f72c9a1b631bc0c1e1b84c/t/5b85fb334fa51a1080348b19/1535507448054/MEICON+Report+Online.pdf

[12] Chang, T.Y. & Wang, S.M., Evaluation of a Nonlinear Finite Element Program – ABAQUS, Report No. AUE-821, Office of Naval Research, 1982.

https://apps.dtic.mil/dtic/tr/fulltext/u2/a128788.pdf

[13] Ramsay Maunder Associates, Multiple Collapse Mechanisms with Identical Collapse Loads, IStructE, The Structural Engineer, August and September Issues, 2016.

https://www.ramsay-maunder.co.uk/blog/rma-contribute-to-the-istructes-and-finally/

https://www.ramsay-maunder.co.uk/blog/the-answer-to-the-istructes-and-finally-question-posed-by-rma/

[14] Angus Ramsay, The Navier Solution for the SSSS Plate, RMA Technical Note, 2018.

https://www.ramsay-maunder.co.uk/downloads/The-Navier-Solution-for-the-SSSS-Plate.pdf