Yield Line Analysis of Reinforced Concrete Slabs: is the 10% Rule Safe?

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Introduction

The yield-line technique is the traditional method of determining the (flexural) collapse load of reinforced concrete flat slabs and steel plates. Most civil/structural engineers will have encountered the technique during their undergraduate education and will be familiar with the concepts. Some may also have used the technique in earnest for the design and/or assessment of an actual slab.

The technique is alluringly simple to apply in that the engineer only needs to conjure up a 'realistic' collapse mechanism comprising a set of yield lines then to calculate the corresponding collapse load by balancing external work as the loads displace and internal work as the yield lines rotate against the plastic moment (calculated from the chosen reinforcement). When the technique is based on work concepts it is often called The Work Method. Limit analysis, such as the yield line technique, usually presents the collapse load in terms of the non-dimensional parameter λ which is known as the load factor and is defined as the collapse load divided by the design load, and should generally be greater than unity.

In terms of limit analysis plasticity theorems the technique is an *upper-bound* approach so that unless the engineer's collapse mechanism is the theoretically correct one, the predicted collapse load will be greater than the theoretically correct value. It is this upper-bound nature, where poorly chosen collapse mechanisms lead to unsafe (non-conservative) predictions of the collapse load, which makes solutions produced by the technique difficult to verify (how unsafe is the collapse load?) and therefore potentially unpopular with those thoughtful engineers who want to sleep soundly at night!

The *Building Research Establishment* (BRE), under The European Concrete Building Project, investigated the strength of reinforced concrete slabs designed using different techniques including the yield line technique. Their findings [1] included the following statement:

> *"Yield line design appears to provide a great opportunity for more competitive concrete building structures, provided the current barriers of lack of familiarity and confidence in its use are overcome. If the opportunity is to be grasped then the concrete frame industry should present designers and the wider construction industry with comprehensive design guidance and design aids to instil confidence in its use." [1]*

In response to the BRE's findings *The Concrete Centre* published *Practical Yield Line Design* [2] as a guide to practising engineers. The guide deals with the issue of potentially unsafe solutions in a pragmatic manner by introducing the *10% Rule*:

> *"A 10% margin on the design moments should be added when using the Work Method or formulae for two-way slabs to allow for the method being upper bound and to allow for the effects of corner levers." [2]*

The 10% Rule is easy to remember and use and, in the sense that most published yield line solutions are no more than about 10% above the theoretical value, has generally served practicing engineers well. However, it is a rule based on empiricism rather than theory and, as with many such rules, counter-examples may be found where the rule is violated. This technical note presents one of a number of known counter-examples to the 10% Rule. The *Landing Slab Problem* was originally considered in 1997 by two experienced engineers exploring improvements to the yield line technique. A yield line solution for this slab was published [3] which, with the benefit of *state of the art* software available today, shows the published collapse load to be over 40% greater than the theoretical value. The provision of such engineering software tools, which can be used with confidence by practicing engineers for arbitrary and novel slab configurations, now enables the BRE's and The Concrete Centre's aim of encouraging more economic design of reinforced concrete slabs through *limit analysis* techniques to be realised.

The Landing Slab Problem

This problem involves a reinforced concrete landing slab typical of the type found in the stairways of modern buildings. It is a two-way slab, in that significant moments are developed in both directions, simply supported on three adjacent sides and the reinforcement at the top and bottom of the slab has equal isotropic moment capacity (m). The slab is loaded with a uniform distributed load (q) as shown in figure 1, and has a unit load to strength ratio (q/m).

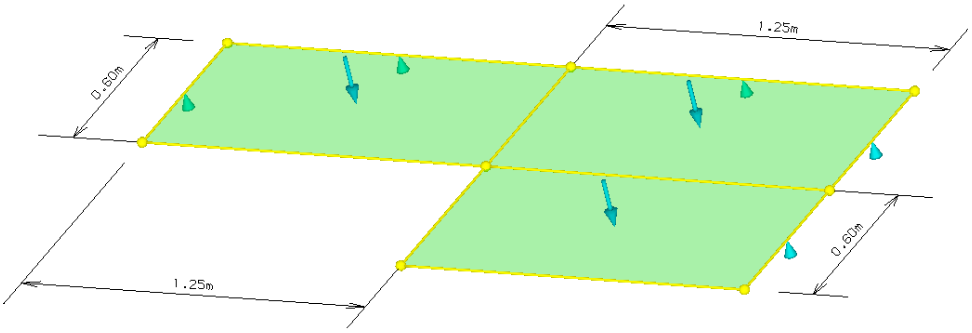


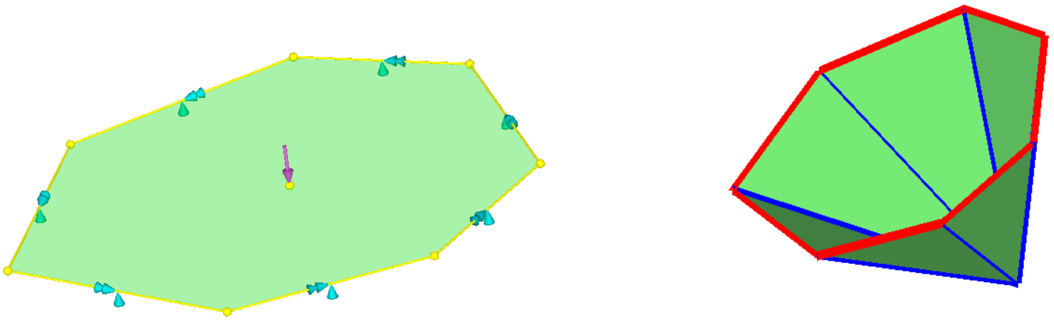
Figure 1: Landing Slab Problem

In terms of the yield criterion, the *Nielsen BiConic Yield Criterion* is often applied to reinforced concrete slabs. The engineer unfamiliar with reinforced concrete may alternatively consider a ductile steel plate for which the *Tresca* or *von Mises* yield criteria would be appropriate. He should note however that, as expressed in [4], for yield criteria other than the Nielsen criterion the yield-line technique cannot capture the exact solution:

> *"However, yield line analysis provides a close estimate, and by successively refining the yield line pattern, the exact solution can be approximated to any desired degree, which is the point. Slabs or plates obeying other yield conditions (eg Tresca or v. Mises) can also be analysed by yield lines, but except for trivial cases the resulting upper bound will never approach the exact solution, however detailed the yield line pattern." [4]*

The Automated Yield Line Technique

The basic concepts of applying the yield line technique may be automated to run on a computer and the *Automated Yield Line Technique* [5], which uses meshes of triangular *rigid* elements, is useful for studying methods for improving the technique. In the automated yield line technique, yield lines can only occur at the interfaces between elements or between elements and boundary lines. Each node that is free to deflect gives rise to a *basic local fan mechanism*, usually involving 'circumferential' hogging yield lines around the boundary of a local patch of elements which all share the node, and 'radial' sagging yield lines. The associated collapse mechanism has the form of an inverted pyramid, or 'pitched roof', as shown in figure 1 where blue lines represent sagging yield lines and red lines indicate hogging yield lines.

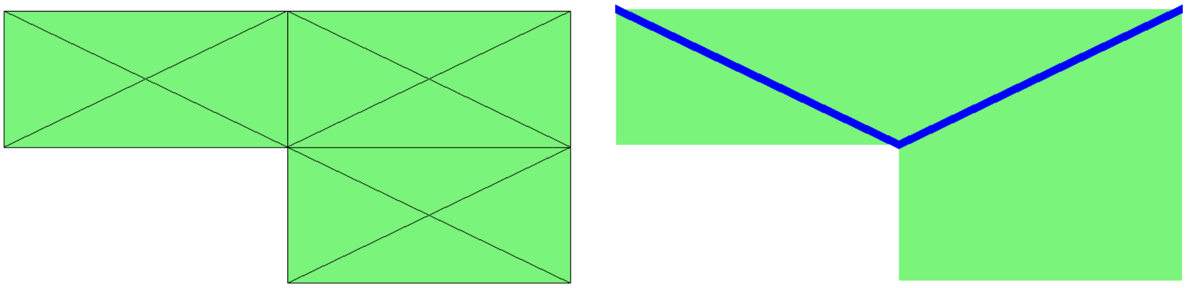


(a) Model

(b) Yield lines and displaced shape

Figure 2: Basic Fan Mechanism

The automated yield line technique, as a limit analysis method, seeks to optimise/minimise the load factor corresponding to all possible linear combinations of basic local mechanisms. The optimisation takes the form of a *Linear Programme.* The meshing algorithm used in the 1997 research produced *Structured Meshes* and the landing problem was divided into three rectangles which were then meshed into four triangles as shown in figure 2a. This mesh contains five basic fan mechanisms. The yield line pattern achieved using this mesh, figure 2b, was considered to be correct since mesh refinement did not produce any change to the solution – more on this below.



(a) Mesh

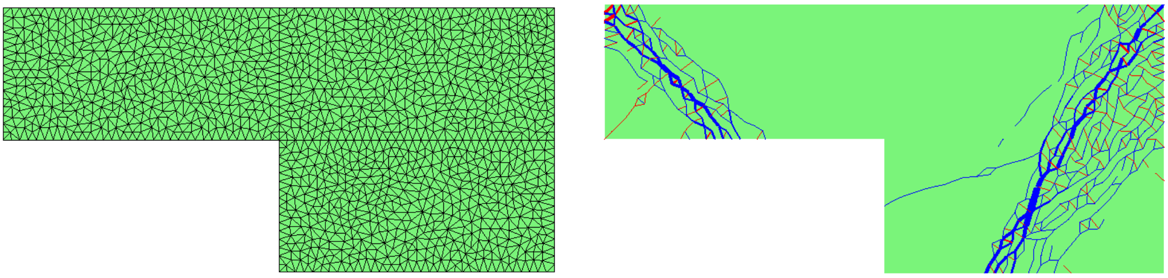
(b) Yield Line Pattern

Figure 3: Results from the 1997 research ($\lambda = 5.86$)

Geometric Optimisation

In 1997 the linear programme was solved numerically using an implementation of the *Simplex Method,* and at that time the algorithmic and computation limitations meant that only fairly coarse meshes could be considered. The software was updated in 2011 by adding routines to produce *Unstructured Meshes* and the simplex method was replaced by the more state of the art *Interior Point Method*. With more efficient software and greater computing power available the landing problem was revisited.

For the 2011 analysis a refined unstructured mesh was chosen as shown in figure 4a. The thickness of the yield lines shown in figure 4b is proportional to the rotation of the yield line so that the thicker yield lines are more dominant than thinner lines. Note also that yield lines with small rotations have been filtered out and are not plotted. If one looks at the yield line pattern with partially closed eyes one may see a new and simpler yield line pattern that might be investigated further. The resulting yield line pattern indicates a different critical collapse mechanism to the one from 1997; whilst the yield lines still start at corners of the slab, the two sagging yield lines which previously met at a slab vertex now exit the slab midway along sides of the slab. This new analysis showed a reduction in load factor of about 7%. It should be noted that the automated yield line technique, like the conventional finite element method, makes use of a mesh of elements. One might thus think, naively as it turns out but not unreasonably, that mesh refinement will lead to the theoretically exact solution. However this is not the case since this will only occur when the mesh comprises element edges that lie on the theoretical yield lines. This is the reason why mesh refinement of the structured mesh of figure 3a produced no change in the solution and misled the researchers into thinking that they had captured the theoretically exact solution.

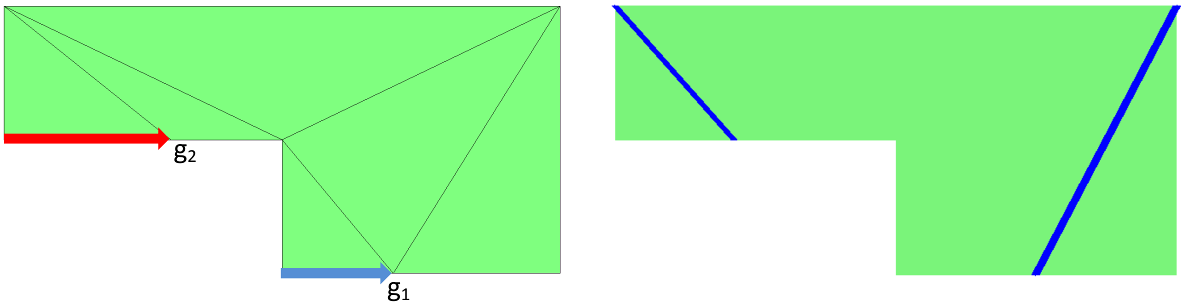


(a) Mesh

(b) Yield Line Pattern

Figure 4: Results from the 2011 Research ($\lambda = 5.47$)

One of the methods for improving the yield line technique considered in 1997 was *Geometric Optimisation* of the yield lines. The idea is rather simple and answers the question of whether the load factor can be reduced by moving the yield lines of a given collapse mechanism. Geometric optimisation was used on the collapse mechanism predicted from the yield pattern of figure 4b. To perform geometric optimisation a simplified mesh is first constructed that can include the collapse mechanism to be optimised. The mesh in figure 5 was selected for this purpose which has two geometric variables as shown. Geometric optimisation led to the yield line solution in figure 5b. This geometrically optimised solution showed a further reduction in load factor of 20%.



(a) Mesh (b) Yield Line Pattern

Figure 5: Results for a coarse unstructured mesh from EFE ($\lambda = 4.38$)

To illustrate how powerful and therefore important geometric optimisation is, the variation in load factor, expressed as a percentage change from the optimised value, is plotted against the value of the geometric variable g_1 shown in blue in figure 5b. This plot is shown in figure 6 which includes insets showing the yield line patterns for four of the values of g_1 (0.25, 0.5, 0.75 and 1.0m) being considered. The first point to note is that beyond a certain value of g_1 (> 0.76m) the collapse mode switches with the right-hand yield line changing from terminating on the bottom edge to terminating at the internal vertex. Clearly any further increase in the value of g_1 leads to no change in the load factor. The second point to note is the huge change in load factor that can be obtained within a single collapse mechanism. In this case the load factor can increase by some 15% - somewhat greater than the 10% suggested in the design guidance.

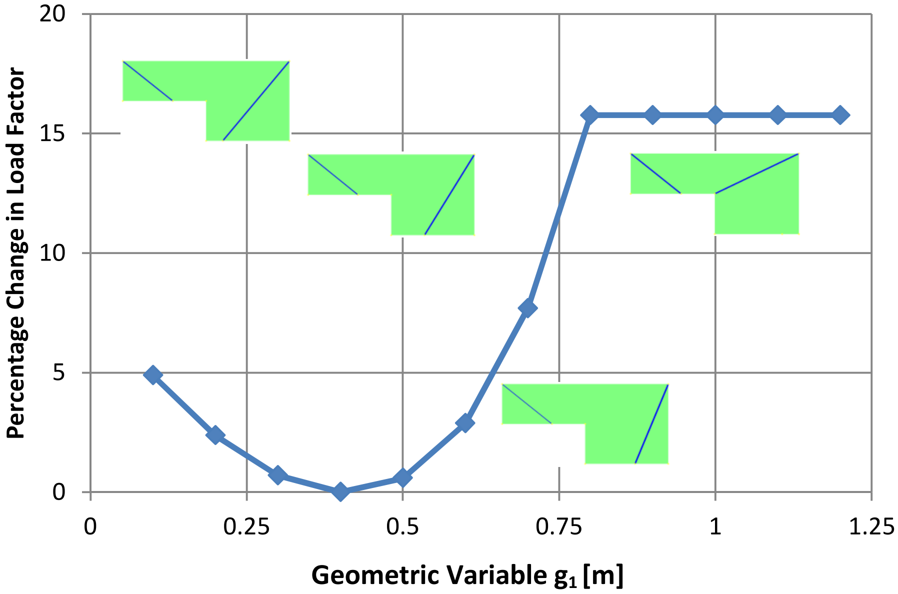


Figure 6: Change in Load Factor with Geometric Optimisation

Using the ideas presented above leads to the suggestion of a general approach for homing in on the theoretically correct collapse mechanism. The first step is to conduct an analysis using a refined unstructured mesh to give a prediction of the correct collapse mechanism. The second step is to construct a coarse mesh that captures this collapse mechanism and to then perform geometric optimisation to minimise the load factor. However, even with appropriate software this approach is far from automated thus requiring intervention and expertise from the engineer in order to achieve reliable results.

Modern Limit Analysis Methods

LimitState, a company 'spun out' of research at the University of Sheffield, have adapted their patented Discontinuity Layout Optimisation (DLO) method to treat reinforced concrete slabs. Although still an upper-bound technique, the DLO method can rapidly obtain solutions that are within +1% of the exact solution.

The results produced by the DLO-based software *LimitState SLAB* [6], come in the form of a yield line pattern which use the same convention as already described for colour and thickness of yield lines - figure 7. This yield line pattern is similar to the geometrically optimised pattern of figure 5 in terms of the dominant yield lines. However it shows additional yield lines that point to a more complicated collapse mechanism for the slab, with more distributed yielding than suggested in figure 5. The load factor from *LimitState SLAB* is 4% lower than that of the geometrically optimised solution already presented.

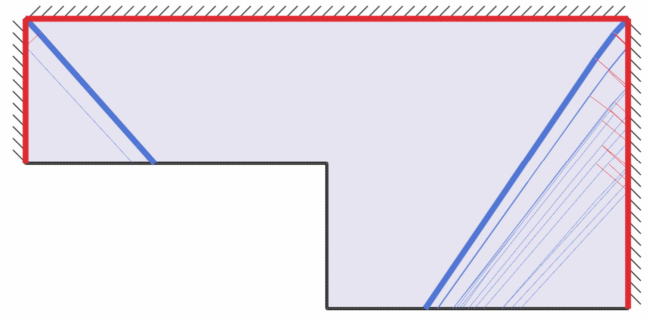


Figure 7: *LimitState SLAB* Solution of 2014 using 4000 nodes ($\lambda = 4.21$)

Since 2011 Ramsay Maunder Associates (RMA), a company set up with the aim of exploiting the ideas of equilibrium in commercial engineering, has been developing a *lower-bound* solver based on equilibrium finite elements (*EFE*) for rapidly predicting solutions that are within -1% of the theoretically exact value [7]. *EFE* is based on the lower-bound theorem of plasticity and, by providing a solution in terms of moment fields that are in equilibrium with the applied loading and that nowhere violate the (Nielsen) yield criterion, produces predictions of the collapse load that are always less than or equal to the theoretically exact value. This is a finite element technique that can converge rapidly with mesh refinement to values within -1% of the true solution.

The results produced by *EFE*, as mentioned, are in the form of moment fields. A simple and effective way of presenting these and, in doing so, demonstrating that these fields do not violate the yield criterion, is to plot contours of the *Utilisation Ratio*. The utilisation ratio can be calculated at points in the model as the amount to which the local moment field can be scaled up before it causes yielding. Such a plot is shown for the landing slab in figure 8 where the contour colours range from zero (blue) to unity (red). The *raison d'etre* of the lower-bound technique is to 'pump up' the moment fields to maximise the load factor and the effect of this can be seen in the contour plot which shows large regions of fully utilised material which is right up to yield. It is noted from figure 8 that yielding occurs in bands around the dominant yield lines as indicated by the less dominant yield lines shown in the solution for *LimitState SLAB* (figure 7).

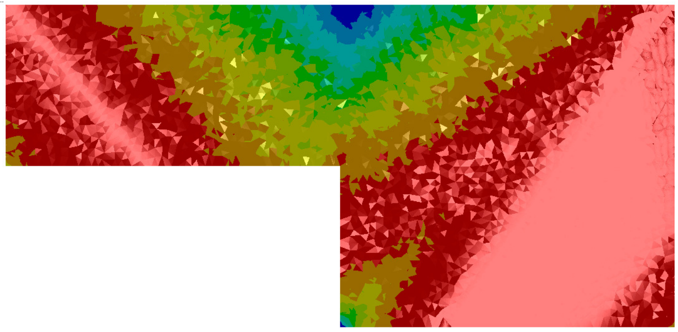


Figure 8: *EFE* Solution of 2014 using 2484 elements ($\lambda = 4.20$)

With upper-bound (*LimitState SLAB*) and lower-bound (*EFE*) solutions to this problem available the theoretically exact collapse load can be predicted within very tight dual bounds:

$$4.20 \leq \lambda \leq 4.21$$

The load factors are within 1% of each other and thus give an extremely accurate prediction of the theoretically exact value. Erring on the side of safety and using the lower-bound load factor in the calculation shows that the published result of 1997 was over 40% too large!

Closure

This technical note has demonstrated, by way of the landing slab counter-example, that the 10% rule advocated in The Concrete Centre's design guidance text as a way of combating the known upper-bound nature of yield line solutions is not always sufficient. Whilst applicable to many published yield line solutions there are others where the over-estimation of the collapse load greatly exceeds the suggested 10%. As shown in this article over-estimates in excess of 40% can occur with rather simple slab configurations. It is perhaps of concern that if even experienced engineers with the luxury of time to undertake research can be led astray; what chance will the practising engineer have when under pressure from his project manager?

The realisation of efficient and robust software for predicting the collapse load of reinforced concrete flat slabs now means that BRE's aim of encouraging engineers to design slabs based on limit analysis techniques can safely be realised and applied with confidence to both conventional more complicated and novel slab configurations.

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Affiliations

The authors of this technical note are directors of companies offering engineering software tools to the practicing engineer. The web sites of these companies are:

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2) http://www.limitstate.com/