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What is Equilibrium Finite Element Analysis?

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Introduction

In the now archaic language of Sir Christopher Wren (1660s) the importance of equilibrium in design is stated thus:

The design must be regulated by the art of staticks, or invention of the centers of gravity, and the duly poising of all parts to equiponderate; without which, a fine design will fail and prove abortive. Hence I conclude, that all designs must, in the first place, be brought to this test, or rejected.

Bill Addis, Building: 3000 years of Design Engineering & Construction, Phaidon, 2007

A more modern and succinct interpretation is provided by Ed' Wilson:

Equilibrium is Essential, Compatibility is Optional

http://www.edwilson.org/book/02-equi.pdf

These statements beg the questions: Why is equilibrium so important and why might compatibility be ignored? The answer to these questions lie in the lower bound theorem of plasticity:

"The only reason why structural designers sleep soundly is the second [lower bound] theorem of plasticity theory. This theorem says that no matter how I designed my structures, they are safe because everything was in equilibrium, nowhere the stresses were too large and I used ductile components and joints."

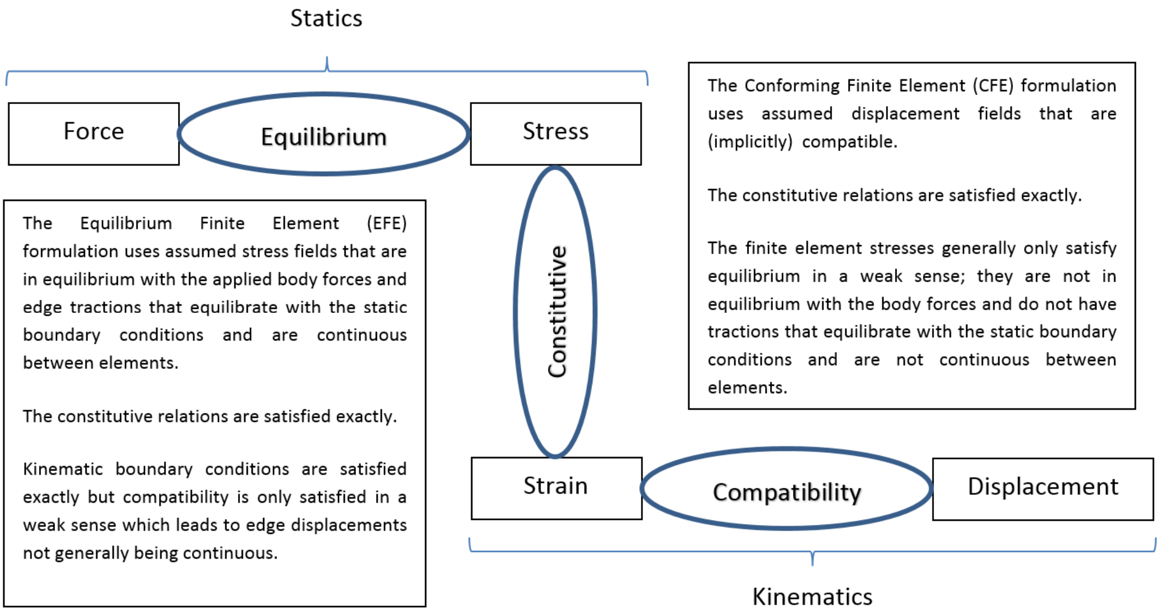
http://homepage.tudelft.nl/p3r3s/CT4150_schedule.html

Thus, if the engineer can invoke the lower bound theorem of plasticity he/she can be satisfied that the design has sufficient strength to take the imposed loads. This requires the engineer's numerical solution to be in equilibrium with the loads and with conventional conforming finite elements (CFE) this is not usually the case. This article, based on [1], presents a different approach which, through an equilibrium finite element (EFE) formulation, offers the engineer an approach which may be used to satisfy the lower bound plasticity theorem.

Finite Element Formulations

Whilst strength of material solutions provide a useful library of theoretically exact solutions for simple practical scenarios, they become less useful when tackling more complicated structures. In these cases the engineer needs to resort to numerical simulation approaches such as the Finite Element (FE) method. The method is approximate but with mesh refinement normally converges to the theoretically exact solution. It is, of course, the engineer's responsibility to ensure that this is actually the case through verification. First the engineer must ensure that the software is capable of producing the correct solution for problems with known theoretical solutions, e.g., strength of materials solutions. This process is known as *software verification* and the engineer should not rely on anecdotal evidence or software vendors own claims that the software is fit for purpose. The faith generated from this exercise enables the engineer to use the software for his/her actual problem which generally has no known theoretical solution. Mesh convergence studies are normally required here to ensure *solution verification* [2].

There are three sets of equations that need to be satisfied for an exact theoretical solution and these are illustrated in Figure 1.



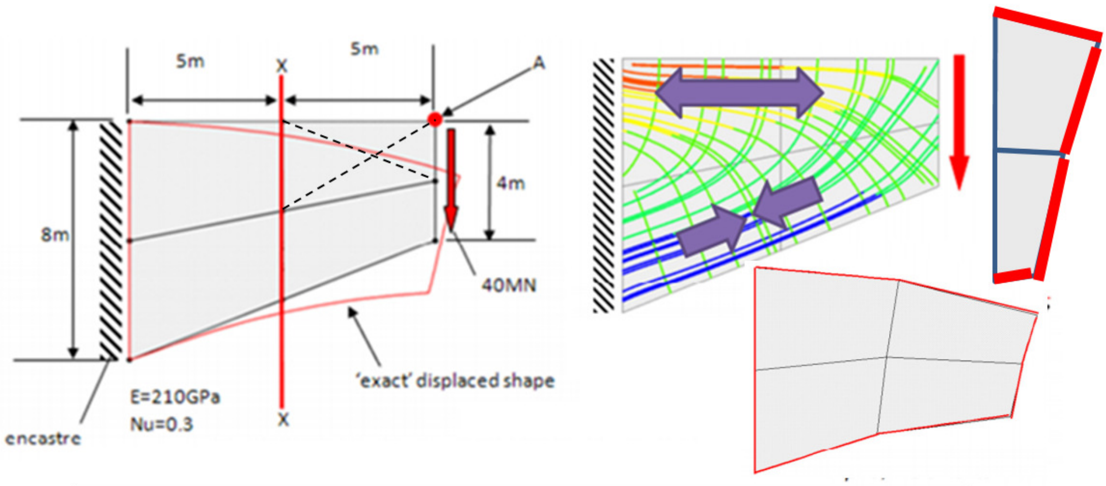
Forces generally include body-forces, applied static boundary conditions which need to be in equilibrium with the stresses and, also, the tractions at inter-element boundaries need to be equal and opposite. The compatibility requirement arises because there are normally more strain than displacement components so that for a given strain field a unique and continuous displacement field can only be determined when the strains satisfy certain compatibility conditions. For conforming displacement elements (CFEs) where continuous displacements are assumed then compatibility is implicitly satisfied. For linear-elasticity the constitutive relations are Hooke's law and for limit analysis they involve an appropriate yield criterion.

Figure 1: Essential relations and weak finite element formulations

Since the FE method is approximate then all three equations cannot be satisfied exactly. Some or all of the equations are only satisfied in a weak sense but as the mesh is refined the weak solution becomes stronger so that, in the limit, as the solution converges to the theoretically exact solution then this weak solution tends to a strong solution. There is a range of possible finite element formulations which approximate the three equations in different manners. Two of these formulations might be termed 'pure' since they only weaken one of the three sets of equations; the remaining two equations being satisfied exactly. Both pure formulations satisfy the constitutive relations and one, the conforming FE (CFE) formulation, also satisfies compatibility whereas the other, the equilibrium FE (EFE) formulation, also satisfies the equilibrium equations.

The CFE formulation, which is used in most if not all commercial FE systems, therefore does not satisfy the equilibrium equations in a *strong* sense. The formulation does however satisfy this condition in a *weak* sense so that as the mesh is refined the approximation becomes less significant and convergence to the theoretical solution is possible. Clearly, if equilibrium is not generally satisfied, then the practising engineer needs to be scrupulous in conducting solution verification if he/she is to obtain an equilibrating stress field from which safe design can be achieved.

In contrast, the EFE formulation satisfies equilibrium in a strong sense whilst weakening the compatibility conditions which then becomes stronger with mesh refinement as the solution converges to the theoretical one. To illustrate the differences between solutions generated by the two pure FE formulations, the linear elastic analysis of the tapered cantilever using a fairly coarse mesh of 2x2=4 elements is considered – Figure 2.



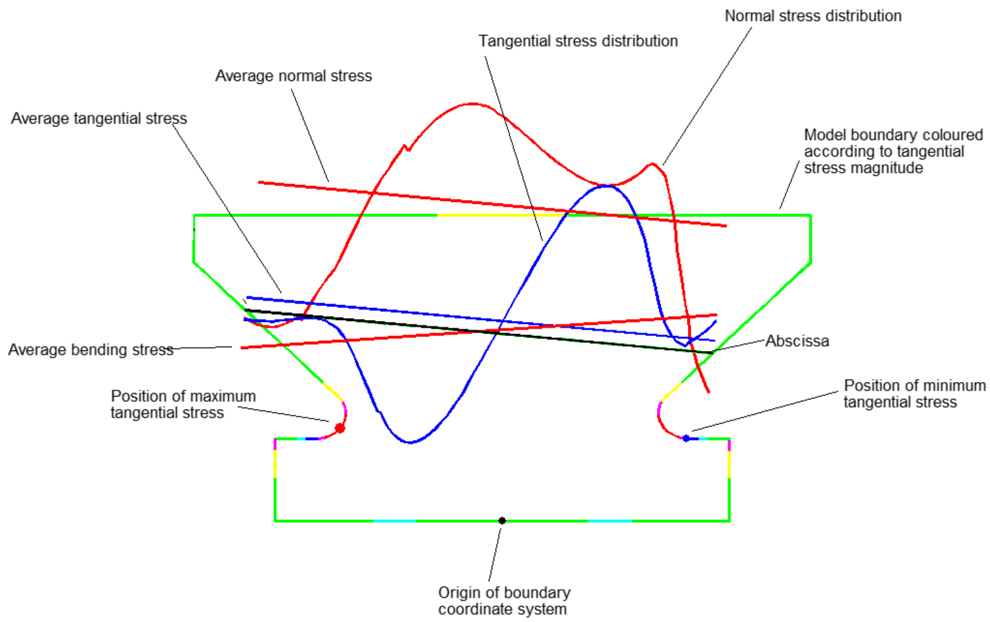
	Displacement (mm)	Normal (MN)		Tangential (MN)		Moment (MNm)	
	A	Left	Right	Left	Right	Left	Right
EFE (*p*=1)	-3.58	0	0	40	40	200	200
CFE (four-noded)	-2.56	27.1	19.7	14.5	80.0	200.2	62.8
CFE (eight-noded)	-3.48	0.0	0.0	39.5	38.9	205.2	220.7
'Exact'	-3.54	0	0	40	40	200	200

Whilst this is a rather simple structure, it demonstrates how the engineer might use the FE results to establish stress resultants across a design section. Both formulations produce FE stress fields that are continuous within elements but discontinuous between elements. Stress linearization is used to establish the stress resultants on the two sides of the section XX. It is seen that whereas the CFE models produce different resultants on either side of the section (neither of which is in equilibrium with the loads), EFE, in this case using linear (*p*=1) stress fields, produces exact values at both sides of the section. The edge displacements for the EFE model are discontinuous as shown in the inset to the figure. Nonetheless, the average value of the displacement is often more accurate than that produced by the CFE, as shown for the vertical displacement at point *A*. It should be noted that the equilibrium quadrilateral element comprises four triangular elements, as indicated in the figure. The elements 'split' the corner allowing the element stresses and therefore the boundary tractions to be discontinuous.

Figure 2: Tapered cantilever problem (linear-elastic)

It should be appealing to the practising engineer that the FE results from an EFE model provides strong equilibrium irrespective of mesh refinement. Although the results will change with mesh refinement, the essential statics of the problem are recovered with even the coarsest mesh and this eliminates one major source of finite element malpractice, i.e., inadequate solution verification, and enables the engineer to concentrate on the essential role of providing a design of adequate strength rather than the esoteric vagaries of the FE method he/she is using.

In a number of industries, codes of practice require the engineer to investigate stress resultants, [3, 4], across selected design sections, e.g., so-called stress classification lines (SCL) in pressure vessel design. With CFE models this is an arduous task since mesh refinement local to the SCL needs to be conducted to ensure accuracy. This requirement is not necessary when using equilibrium elements so that the engineer can ignore the mesh and simply place an SCL wherever it is desired. An example of the output of a software tool written for this purpose is shown in Figure 3.



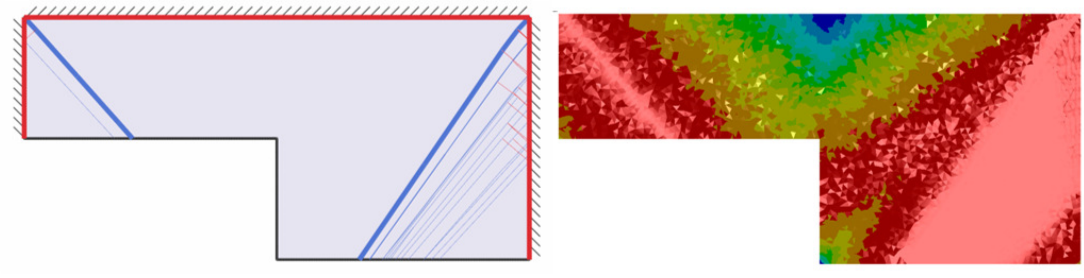
The example is of a turbine blade root with stress and resultant distributions shown for the section defined by the abscissa. With this sort of software it is a rather simple task to conduct automatic optimisation so that the termination points of the design section are moved around the boundary to establish, for example, the section with the maximum stress resultant.

Figure 3: Equilibrating stress resultants on a section of an EFE model

The practising engineer may need to deal with strength calculations both in terms of *assessment* of an existing structure, say for a change of load duty, and *design* of a new structure. Detailed design is essentially a process where the engineer needs to optimise the structural *capacity* to cope with the *demand* in strength. In reinforced concrete (RC) design, the capacity would be controlled by the reinforcement layout and sizing. Whilst the design problem can be programmed into effective software tools, there are few generalised design tools available and so the engineer will often approach this through iterative *design-by-assessment*; for RC design, an initial reinforcement layout is chosen and then modified accordingly to provide the appropriate strength capacity based on the results from the previous assessment.

In assessment and design the engineers task of providing sufficient capacity is greatly simplified if a complete picture of the way in which the applied loads are transmitted to the supports, i.e., the load path, is available. The results from an EFE model can provide this through principal stress or moment trajectories, [5]. The tapered cantilever problem of Figure 2 includes the principal stress trajectories which may be used to formulate a strut & tie model, as indicated, so that the required reinforcement may be established.

When the engineer considers the strength of a structure or component, then, provided the material is ductile, it is the plastic limit load (load to cause plastic collapse) rather than the elastic limit load (load to cause first yield) that defines the strength. Limit State Design uses the plastic collapse load as one of the Ultimate Limit State (ULS) conditions and the use of plastic methods of analysis is embodied in modern codes of practice such as the EuroCodes for RC (EC2) and steel design (EC3). Limit analysis provides a method of obtaining the plastic limit load based on a rigid, perfectly plastic material model. It is generally assumed to be a conservative approach since such phenomena as strain hardening and membrane effects, normally considered to have a strengthening effect, are not considered. Limit analysis is based on the plasticity theorems of which there are two. The first involves determining kinematically admissible collapse mechanisms and if the mechanism is not an optimal solution the method produces an *unsafe* upper bound to the theoretical collapse load. The second involves determining equilibrating stress/moment fields and if the stress field is not optimal then the method produces a *safe* lower bound to the theoretical collapse load. Modern software tools for the limit analysis of RC slabs were presented in [6] and the advantage of using both tools, in a dual analysis approach, as illustrated in Figure 4, was discussed in order to provide a verified collapse load.

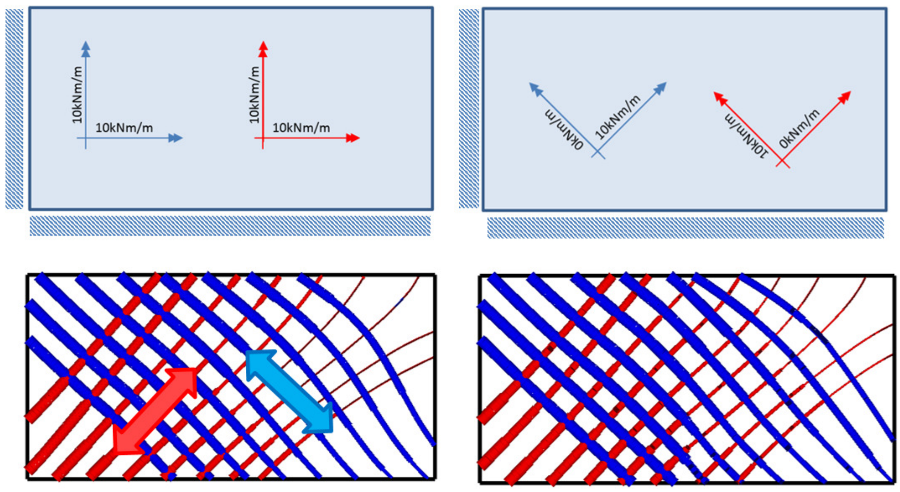


(a) CFE ($\lambda = 4.21$) (b) EFE ($\lambda = 4.20$)

The CFE result is an upper bound solution obtained through a modern yield line software tool whereas the EFE result is a lower bound solution.

Figure 4: Limit analysis solutions for an RC landing slab, [6]

Whilst the CFE model, through yield line patterns, provides a good representation of the collapse mechanism, no useful information is provided for the moment fields. This is only available through an EFE approach. Thus, in design-by-assessment EFE is to be preferred since the complete statics can be visualised through the principal moment trajectories and the collapse load is safe irrespective of the level of mesh refinement. The uniformly loaded, simply supported RC slab configuration shown in Figure 5 illustrates how, with a complete picture of the statics, the engineer can very easily make significant economisations in the reinforcement required for a given slab capacity.

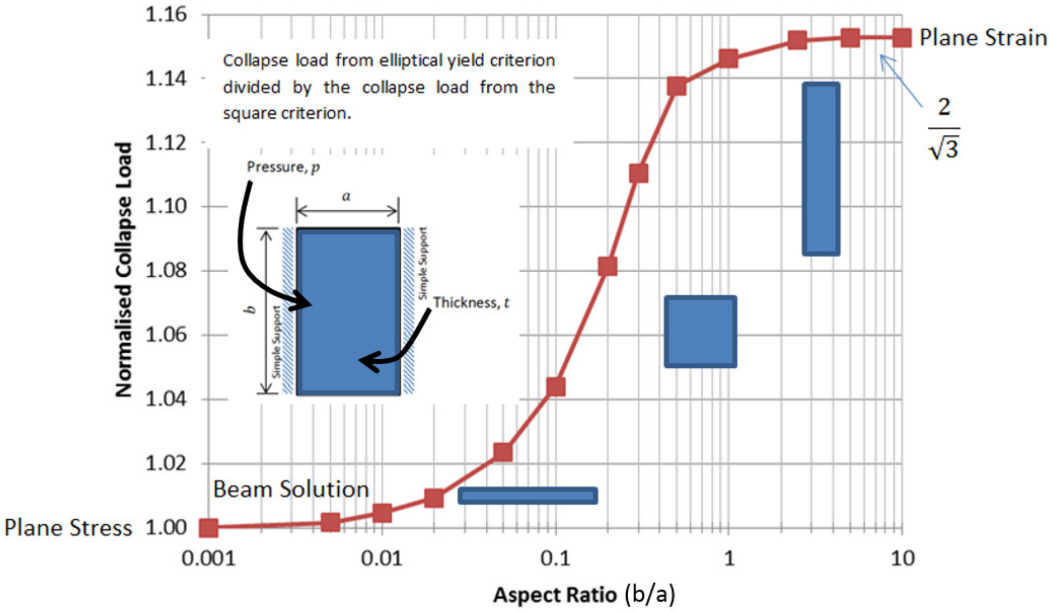


23.59kPa 23.30kPa

The figures in the left column show an initial isotropic reinforcement layout from which the collapse load for the slab may be determined. The principal moment trajectories for this layout show the optimal directions for the reinforcement layout (hogging in red and sagging in blue) and it is easily seen from this figure that by rotating the reinforcement through 45 degrees it is possible to eliminate 50% of the steel. This is shown in the right column with the reduction in collapse load being minimal.

Figure 5: Lower bound solutions showing principal moment trajectories and collapse loads, [7]

In addition to the limit analysis of RC slabs, which requires a square yield criterion, EFE may also be used for safe prediction of the collapse of steel plates based on the von Mises yield criterion. This recent development of EFE has enabled a review of the published data offered to practising engineers for the plastic limit load of steel plate configurations. It has been found that much of this data is erroneous and either based on approximate elastic solutions or yield line analysis. There is a fundamental flaw in applying yield line analysis, which implicitly used the square yield criterion, to materials where yield is more appropriately represented by the von Mises criterion. The results shown in Figure 6 are for the plastic collapse of a uniformly loaded, simply supported rectangular plate.



The plastic collapse load from yield line analysis, which implicitly adopts the square yield criterion, is invariant with plate aspect ratio. A more appropriate yield criteria for ductile steels is the von Mises or elliptical criterion and if this is adopted then the capacity of the plate increases by up to 15%. In the context of simulation governance, [2], this is an example of a situation where the mathematical model, in terms of the yield criterion, adopted implicitly in yield line analysis is incorrect and, thus, attempts at validation might well be thwarted. The plastic limit loads for common plate configurations as determined by EFE will be published later this year, [8], to provide an updated and reliable data set for use by practising engineers.

Figure 6: Plastic limit load for a rectangular plate configuration, [8]

Closure

This short article has been written with the aim of illustrating the essential characteristics of the EFE formulation. The authors' feel that this formulation provides some distinct advantages over the conventional CFE approach particularly when the practising engineer is involved in design/assessment for structural strength. Whilst this article has only scratched the surface of equilibrium finite elements, further information about may be obtained at http://www.ramsay-maunder.co.uk/ and in March of this year a text detailing the theoretical background to this FE formulation will be published, [9]. A more detailed some of the theoretical formulations upon which EFE is based and a less theoretical text is to be prepared during 2017 as part of the NAFEMS, 'Why do?' series of booklets.

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