Visualisation of stress fields - from stress trajectories to strut and tie models.

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Introduction

Structural designers generally need to appreciate how loads are transmitted through structures by following paths of internal forces and/or stresses which are statically admissible. Paths of principal stresses, or “stress trajectories”, were developed by Maxwell from photoelastic techniques around 1850, and by Culmann in the context of graphical methods in 1866 [1]. Stress trajectories continued to be used in photoelasticity in its heyday [2] before computational techniques became predominant. Postprocessing stages of finite element analyses now produce graphical forms of stress fields – generally in the form of contour maps of stress components, von Mises stress, or vector plots of principal stresses as discrete crosses at the centres of elements. The arms of the crosses have lengths proportional to stress magnitudes, and this can lead to scaling problems when a complete overview of the structure is required. Recent developments in the design of reinforced concrete structures as continua focus on internal force paths as “strut and tie” models [3]. In forming such models guidance can be given by knowledge of stress trajectories. This paper reports on recent work to integrate developments in hybrid equilibrium elements with improved methods of visualising stress fields, and transforming such fields to similar but simplified strut and tie forms.

Discrete crosses

For overall views of stress fields conventional plotting of principal stress crosses generally suffers from scaling problems, i.e. a uniform scale can lead to lines too short to see to lines too long which overlap in a confusing way. Discrete crosses are very simple to compute, and their visualisation can be improved by fixing the lengths of the arms and specifying other attributes such as colour and/or width to convey the relative magnitudes of the stresses. An example is shown in Figure 2, which represents a residual, i.e. hyperstatic, planar stress field in a triangular prism. Contragredient components of stress are described by the 5th degree polynomials in Equation (1) using the oblique axes in Figure 2.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{Bmatrix} = \begin{Bmatrix} \dfrac{x^3}{b} - \dfrac{3x^2y}{a} - \dfrac{2x^4}{b^2} + \dfrac{12x^2y^2}{a^2} + \dfrac{x^5}{b^3} + \dfrac{3x^4y}{ab^2} - \dfrac{6x^3y^2}{a^2b} - \dfrac{10x^2y^3}{a^3} \\ \dfrac{3xy^2}{b} - \dfrac{y^3}{a} - \dfrac{12x^2y^2}{b^2} + \dfrac{2y^4}{a^2} + \dfrac{10x^3y^2}{b^3} + \dfrac{6x^2y^3}{ab^2} - \dfrac{3xy^4}{a^2b} - \dfrac{y^5}{a^3} \\ \dfrac{-3x^2y}{b} + \dfrac{3xy^2}{a} + \dfrac{8x^3y}{b^2} - \dfrac{8xy^3}{a^2} - \dfrac{5x^4y}{b^3} - \dfrac{6x^3y^2}{ab^2} + \dfrac{6x^2y^3}{a^2b} + \dfrac{5xy^4}{a^3} \end{Bmatrix} \quad (1)$$

For an equilateral triangle the stress field is skew-symmetric about the centre line shown in Figure 2. The grid of points is arranged in a similar triangular pattern with a spacing equal to a 45th of the side length. The lines are colour coded blue for tension and red for compression, and with line widths proportional to the stress magnitude. Any lines with stress less than 5% of the maximum magnitude have been omitted to simplify the plot. The selection of grid spacing is a compromise between having enough information to be meaningful but not so much as to be meaningless! This type of plot is useful in guiding the development of continuous trajectories. Furthermore, a strut and tie model becomes evident with two main internal “nodal zones” as shown in Figure 3.

Continuous trajectories

Governing equations

A fundamental theorem on stress trajectories states [2] that a system of stress trajectories can be divided into two orthogonal families of curves for which the tangents to one family give the directions of the major principal stress, and the tangents to the other family give the directions of the minor principal stress. The principal stress difference (major - minor) remains positive everywhere and along all trajectories except at points/lines/areas where the stress is the same in all directions. Such points are termed isotropic.
The Lamé-Maxwell differential equations of equilibrium for plane stress can be formed with reference to curvilinear axes which coincide locally with the stress trajectories. Figure 1 illustrates stresses on a curvilinear infinitesimal element.

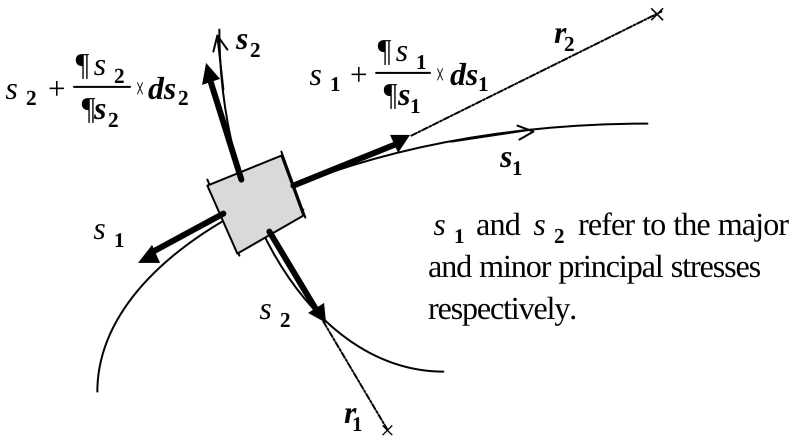

Figure 1: Principal stress components using curvilinear coordinates.

In the absence of body forces, the statically admissible stress fields satisfy equilibrium equations of the form:

$$\left(\sigma_1 - \sigma_2\right) = \rho_1 \cdot \frac{\partial \sigma_2}{\partial s_2} = \rho_2 \cdot \frac{\partial \sigma_1}{\partial s_1} \tag{2}$$

The local curvature of a trajectory can thus be determined from estimates of local stress gradients and differences between principal stresses. The additional constraint for compatibility of elastic strains for a homogeneous isotropic material is:

$$\nabla^2\left(\sigma_1 + \sigma_2\right) = 0 \tag{3}$$

In other words, the sum of the principal stresses must be harmonic in order to satisfy equilibrium and compatibility and hence be a Trefftz stress field.

Numerical procedures and problems

Two aspects of plotting trajectories can lead to numerical problems. Firstly, isotropic points can be of two types, non-interlocking (type 1), or interlocking (type 2). Both types of isotropic points give rise to singularities in the curvatures of the trajectories i.e. a curvature tends to become infinite in the neighbourhood of an isotropic point, the singularity appears stronger for type 2 where trajectories do U-turns about the isotropic point. The other problematic feature is the possibility of spiralling away from a closed orbit onto another one due to small rounding errors, or spiralling onto a fixed orbit at a slow rate of convergence. The latter problem is affected by the stress field in the neighbourhood of the orbit and rounding errors in the numerical procedures. Both Euler and Runga Kutta methods have been used in the context of stress visualisation as well as flow visualisation [4,5].

Examples of continuous trajectories

The residual stress in a triangle problem, as illustrated in Figures 2 and 3, is used again as an example. Now continuous trajectories are developed using 4th order Runga-Kutta method with a step length equal to one quarter of the grid spacing. Trajectories are initiated at a single grid point and are formed in all four possible directions. Termination of trajectories depends on three criteria: (i) a boundary is reached in a direction normal to it; (ii) an isotropic point is recognised by a small difference between principal stresses; (iii) an upper limit is placed on the number of steps taken. Figures 3 to 5 show plots from three different points marked ✲ , with one trajectory of each plot terminating at the isotropic point (type 2) of zero stress at the centroid of the triangle. Furthermore spiralling occurs with convergence onto one or two closed orbits from within or without these orbits. The discontinuous and continuous plots give complementary information. The strut and tie model helps to explain the significance of the fixed orbits, the material within them forms “nodal zones” with only normal stresses acting on the boundaries of these zones. In essence each node has three bands of tension or compression acting normal to the node. The interesting property appears to be that these two orbits are unique within the triangle.

The second example is of a tapered cantilever supporting a udl along part of its top edge. This involves a straightforward load transmission between the top edge and the supported edge in the form of a compression band, and tensile stresses concentrated mainly in the top zone of the cantilever. Colour coded continuous trajectories are plotted in Figures 6 and 7 based on Euler’s method, and a well defined simple strut and tie model is superimposed.

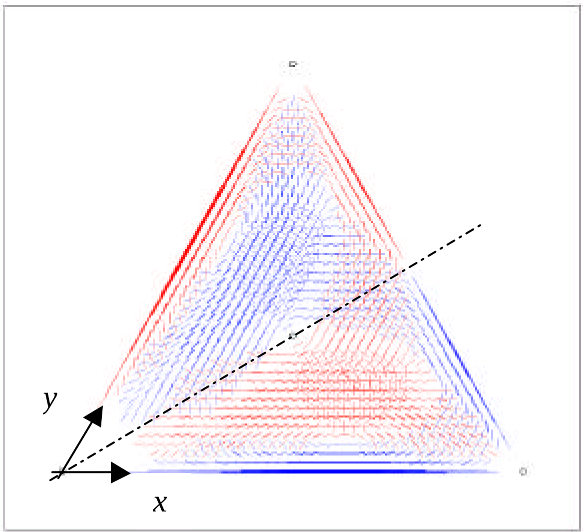

Figure 2: Discrete crosses for residual stresses

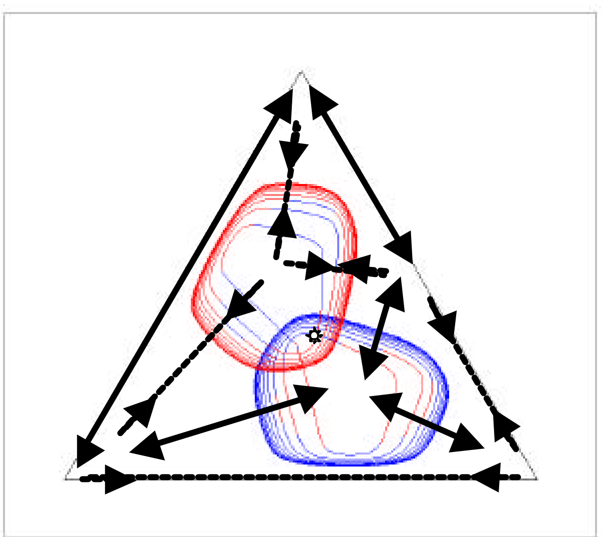

Figure 3: Strut and tie model with nodal zones

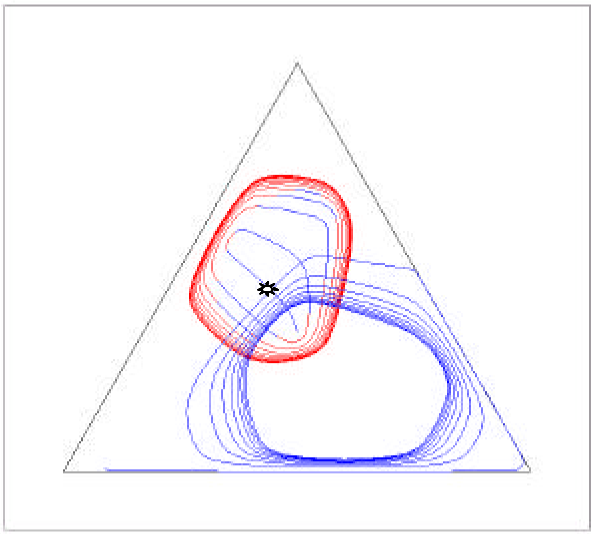

Figure 4: Trajectories from point .

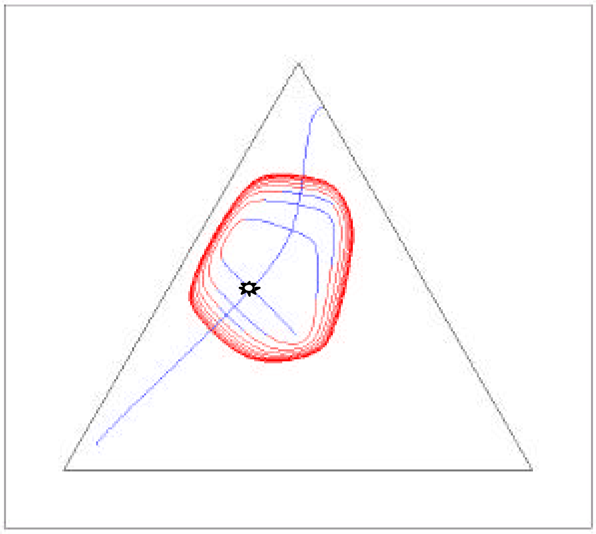

Figure 5: Trajectories from point .

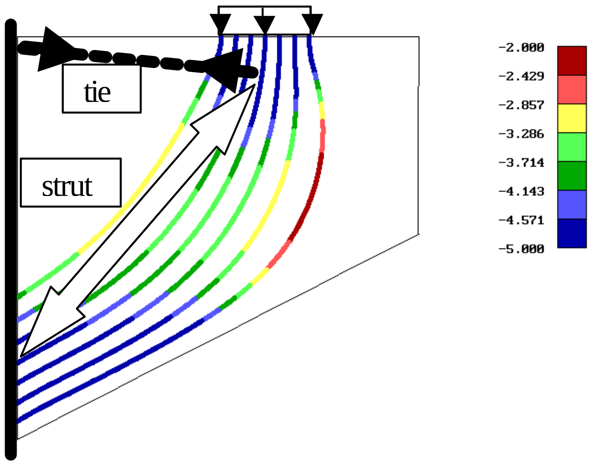

Figure 6: Compressive trajectories

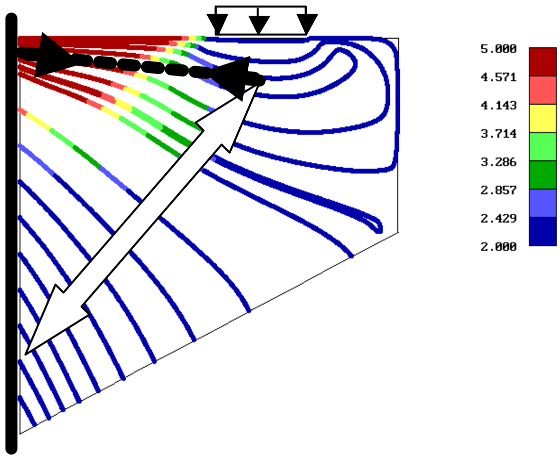

Figure 7: Tensile trajectories

Closure

The 4th order Runga-Kutta with appropriate criteria for terminating plots appears to be robust in that it produced expected results for the 5th degree problem of a hyperstatic stress field. This problem contains all the problematic features for plotting continuous trajectories, and hence is considered as a good benchmark problem. Further work is required to “clean” the spirals which can interfere visually with other trajectories, and to develop algorithms which will help to transform trajectory plots into simplified but similar strut and tie models of potential use to the designer of reinforced concrete, or more general forms of composite structure.

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