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An Error in Timoshenko's 'Theory of Plates and Shells'

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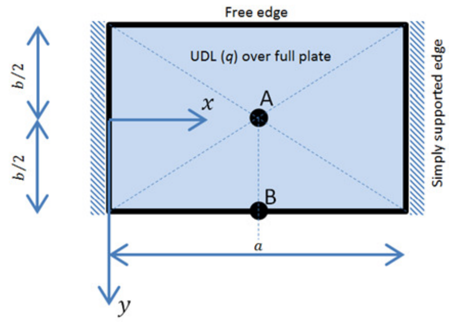
The authors recently conducted a study into the elastic behaviour of thin (Kirchhoff) plates using commercial finite element (FE) software. In attempting to verify the FE solution it was compared to results presented in Timoshenko's text [1] and a significant difference was observed. This article presents the work conducted to uncover the reason for this difference and reveals an error (probably typographical) in the text. The source of the error is identified and it is demonstrated how such errors might propagate into other texts on the subject of plates. The significance of the error to the practising engineer is also discussed.

The Plate Configuration Considered

The plate considered is rectangular with an aspect ratio b/a. It is simply supported on two opposite sides and loaded with a uniformly distributed load (UDL) as shown in figure 1.



b/a	Point A $x = a/2\,;\, y = 0$			Point B $x = a/2\,;\, y = \pm b/2$	
	$w = \alpha_1 \frac{qa^4}{D}$	$M_x = \beta_1 qa^2$	$M_y = \beta_1' qa^2$	$w = \alpha_2 \frac{qa^4}{D}$	$M_x = \beta_2 qa^2$
	α_1	β_1	β_1'	α_2	β_2
0.5	0.01377	0.1235	0.0102	0.01443	0.1259
1.0	0.01309	0.1225	0.0271	0.01509	0.1318
2.0	0.01289	0.1235	0.0364	0.01521	0.1329
∞	0.01302	0.1250	0.0375	0.01522	0.1330

Figure 1: Plate configuration and Timoshenko's results

This problem is considered in Article 48 (p214) of reference [1] and the deflections and moments at points A and B are reported in the text (table 47, p219) for a Poisson's ratio of $\upsilon = 0.3$. This table has been reproduced in figure 1 where *D* is the flexural rigidity of the plate and *w* is the transverse displacement.

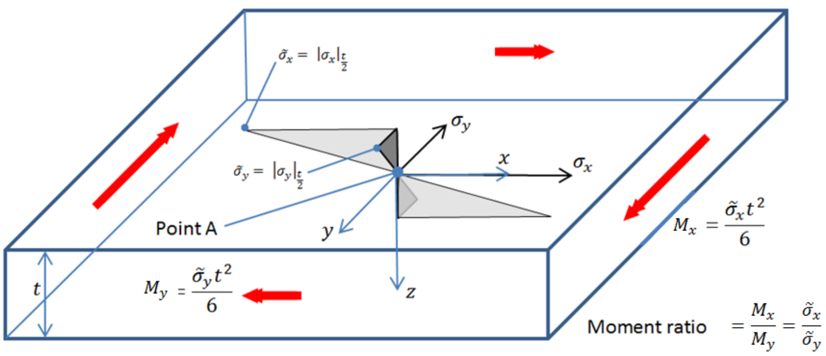


Figure 2: Moments and stresses at centre of plate (point A) and definition of moment ratio

In figure 2, an infinitesimal region around the centre of the plate, is shown, together with the moments and stresses. The uniformly distributed load causes sagging moments in both longitudinal and transverse directions which induce stresses in the plate, linearly distributed across the thickness, as shown with the stresses on the top surface both being compressive. Note that the moment m_x causes a direct stress in the x direction (σ_x).

Finite Element Analysis of Plate

The aspect ratio of the plate considered was 0.5 and the authors chose to study the convergence of the moment ratio (defined in figure 2) with both mesh refinement and span to thickness ratio (a/t). The reason for considering convergence with span to thickness ratio was that the finite element system used only provided thick (Reissner-Mindlin) plates, and it was therefore necessary to ensure that the chosen thickness was small enough to have removed the influence of shear deformation which is not considered in the thin (Kirchhoff) formulation which is being investigated.

The results from this convergence study are summarised in the figure 3 where an initial mesh of 1x2=2 elements was used with uniform mesh refinement and span/thickness ratios between 2 and 2000 were studied. Both studies converge to a moment ratio of 10.19 as shown in the figure.

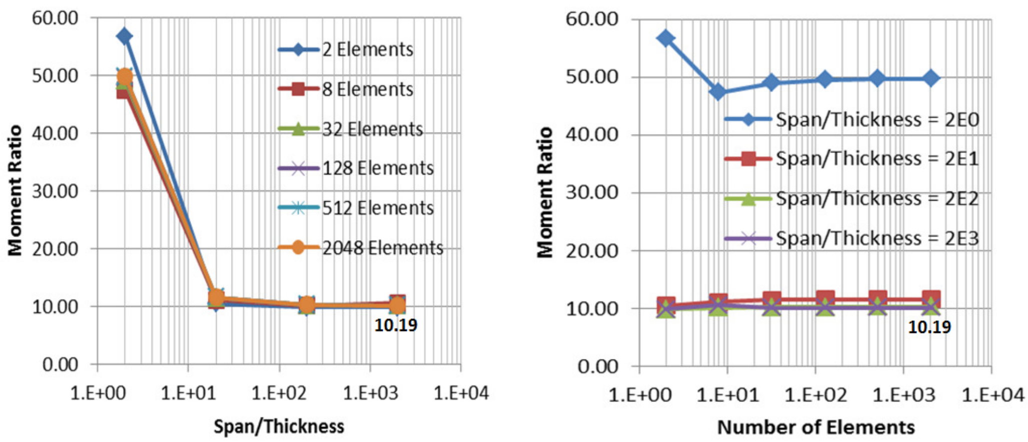


Figure 3: Convergence of moment ratio at point A with span/thickness ratio and mesh refinement

The ratio of the moments presented in [1] is 12.11, so there is a significant difference, approaching 20%, between the FE values and the published moments and this needs further investigation.

Development of a Computer Program

Timoshenko's text provides an expression for the plate transverse displacement (w) as a single series (attributed to Levy and presented on p217, eq(h)) which may be twice differentiated to produce the moments – see p39 of reference 1 for example. The relevant equations are shown in the appendix to this article together with the subroutine used for evaluating some of the results presented.

The expressions for the moments were coded into a small program so that they could be evaluated at a given point within a plate of arbitrary aspect ratio (b/a). The summation implied by series solution is carried out in a loop for which only odd indices are considered and the upper value of the index is maintained as a variable in the program. Rapid, oscillatory, convergence is observed with the moment ratio for 26 terms in the series being 10.17843 as shown in figure 4.

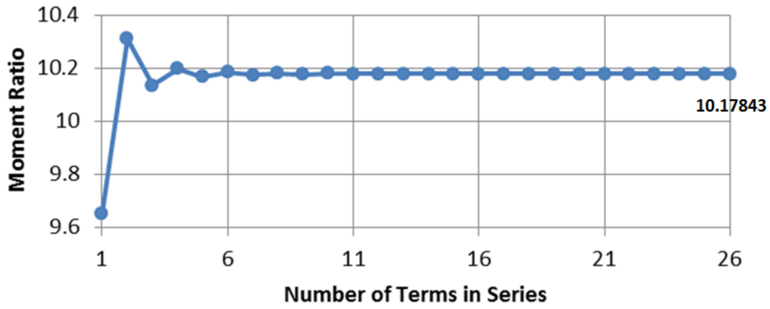


Figure 4: Convergence of moment ratio at point A

The program produces values of displacement and moment at any point. These values may be used to plot distributions across the plate, and inspection of these distributions for satisfaction of the kinematic and static boundary conditions will provide verification that the program is correct.

The displacement field, not shown in this article for conciseness, demonstrates that the zero displacement condition along the simply supported edges is satisfied and that the field possesses the expected symmetry about the lines $x = a/2$ and $y = 0$.

The Cartesian components of moment are shown in figure 5. The static boundary conditions require there to be zero bending moment along all edges and this is clearly seen. The torsional moments are not required to be zero along the boundary, as Kirchhoff theory is assumed, but they should be zero along the two lines of symmetry and this is seen to be the case. The principal moments and the von Mises moment field, M_{vM}, are shown in the figure and it is seen that the point of first yield is at point B, the centre of the free edges.

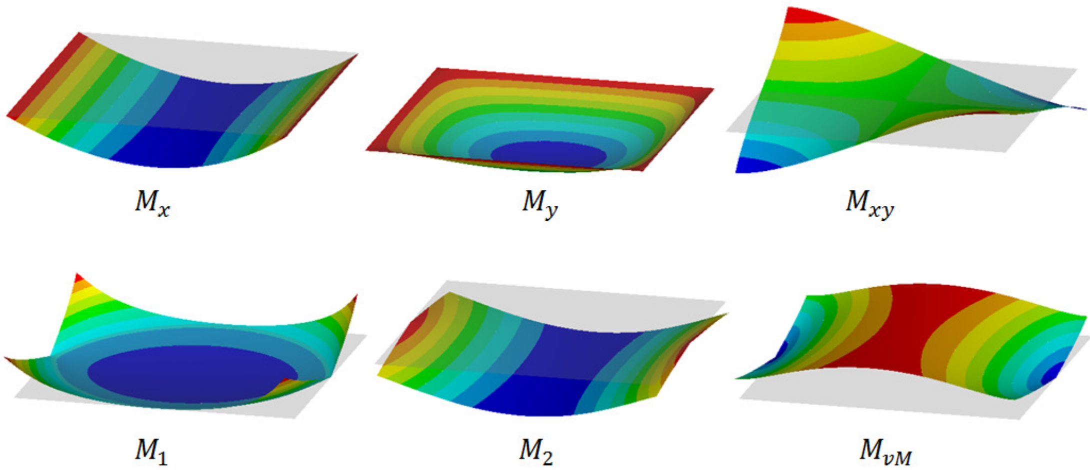


Figure 5: Moments from program

An additional finite element result was produced using a pure Kirchhoff finite element. This gave a moment ratio of 10.1784 which, to four decimal places, is identical to that produced by the program thus independently verifying the program. The moment ratios from the four independent sources considered are shown in table 1.

Source	Moment Ratio at Point A
Timoshenko	12.11
FE (Reissner-Mindlin)	10.19
Program	10.18
FE (Kirchhoff)	10.18

Table 1: Summary of moment ratios at point A from different sources

The results shown in the table indicate that there is something amiss with the values published in Timoshenko's text, at least for an aspect ratio of 0.5, and further investigation of the individual moment components used in the moment ratio show that it is the value of M_y at point A which is in error with the value in the text being 0.0102 and the value from the program being 0.0122.

The table of point results produced in Timoshenko (and reproduced in figure 1) is attributed to a 1936 publication by D. L. Holl [2] studying the problem presented in this article. D.L. Holl's paper presents results, such as the moments, for this plate in terms of tables of data which is already summed for use in the appropriate equation. The summation involves only three terms in the series (which from figure 4 is not likely to be particularly accurate) and in order to provide the data in Timoshenko's table, further calculations are required. The difference between the published values and those produced by the program is reasonably small for all but M_y for an aspect ratio of 0.5 as shown in table 2. As such, the error in Timoshenko's text appears not to be due to Holl.

b/a	Point A			Point B	
	$x = a/2\ ; y = 0$			$x = a/2\ ; y = \pm b/2$	
	$w = \alpha_1 \frac{qa^4}{D}$	$M_x = \beta_1 qa^2$	$M_y = \beta_1' qa^2$	$w = \alpha_2 \frac{qa^4}{D}$	$M_x = \beta_2 qa^2$
	α_1	β_1	β_1'	α_2	β_2
0.5	0.41	-0.12	-16.03	-1.47	-1.50
1	-0.03	-0.04	0.08	0.52	0.54
2	0.02	0.03	0.03	0.05	0.07
∞	-0.01	-0.05	-0.05	0.01	-0.01

Notes: (i) Percentage differences calculated as 100 x (book – program)/program (ii) The values in the table converge very rapidly with increasing aspect ratio and the value used for the 'infinite' aspect ratio was 10.

Table 2: Percentage difference in displacements and moments

As already noted, the values of M_y at point A for the book and program are, respectively, 0.0102 and 0.0122. It is interesting to surmise that there is a typographical error in the book value, since if the last two digits are transposed then it becomes 0.0120 and the error reduces to -1.64% which is much more in line with the error in the other values reported in table 2.

Closure

This article has uncovered, by chance, an error in the published result for the transverse moment at the centre of the plate configuration considered when the aspect ratio is 0.5 and it illustrates the sort of care required by practising engineers when taking published data at face value, even when it comes from such revered texts as reference [1]. With the wide availability of finite element systems, the practising engineer can, and should, check the values he or she is going to use in the design or assessment of a structural member. It is also interesting to note the fact that published errors can propagate. In this case erroneous data published in 1936 was still being used in the 28th reprint of Timoshenko's text published in 1989 and also appears in Szilard's 2004 publication on the theory and analysis of plates [3] (case number 103). The authors of this current article have contacted the publishers of [1] regarding this error asking whether it might be corrected at a future reprint. However it is understood that no further reprints are likely. This raises the question of how one then might protect practising engineers against the propagation of erroneous published data? One way to do this would be to have an online repository of such errors which engineers can access to check that there are no reported errors in the data they are proposing to use. In the absence of such a facility the best one can do is publish the finding, as here, with the hope that it will reach the intended audience.

With regard to the engineering significance of this finding, the error leads to an under-prediction of the minor (transverse) component of the moment at the plate centre. The engineer designing a steel plate might use the moments to calculate the von Mises moment and ensure that this is below the yield moment for the steel being used. Since the von Mises moment is greater at the centre of the free edge (point B) than at the centre, then provided the engineer notices this, the erroneous value in the table would never be used. For a designer of a reinforced concrete slab however this number may well be used to size the reinforcement lying parallel to the y axis and an under-prediction of some 16% might lead to a situation where the structure is pushed out of the elastic region and into the plastic region. The degree to which this will occur should however be well within the ultimate capacity of the slab but may be undesirable in terms of serviceability issues such as cracking of the concrete.

Acknowledgements

The authors would like to thank:

Professor J. P. B. M. Almeida of the Department of Civil Engineering & Architecture, IST, University of Lisbon, Portugal for running the plate through his Kirchhoff equilibrium finite element software.

Professor Husain Jubran Al-Gahtani of the King Fahd University of Petroleum & Minerals, Dhahran, in the Kingdom of Saudi Arabia for independently verifying the results produced by the program using Mathematica.

References

[1] S.P. Timoshenko & S. Woinowsky-Krieger, 'Theory of Plates and Shells', 2nd Edition, McGraw-Hill International Series, 28th Printing 1989.

[2] D.L. Holl, Iowa State College, Engineering Experiment Station, Bulletin 129, 1936.

[3] R. Szilard, 'Theory and Application of Plate Analysis', Wiley 2004.

Appendix: A FORTRAN Program

In Timoshenko's text, reference is made to Levy's single series solution for the transverse displacement:

$$w = \frac{qa^4}{D} \sum_{m=1,3,5,\ldots}^{\infty} \left(\frac{4}{\pi^5 m^5} + A_m \cosh \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}$$

where:

$$A_m = \frac{4}{\pi^5 m^5} \cdot \frac{\nu(1+\nu)\sinh\alpha_m - \nu(1-\nu)\alpha_m \cosh\alpha_m - m\pi\lambda(2\cosh\alpha_m + \alpha_m \sinh\alpha_m)}{(3+\nu)(1-\nu)\sinh\alpha_m \cosh\alpha_m - (1-\nu)^2\alpha_m + 2m\pi\lambda\cosh^2\alpha_m}$$

$$B_m = \frac{4}{\pi^5 m^5} \cdot \frac{\nu(1-\nu)\sinh\alpha_m + m\pi\lambda\cosh\alpha_m}{(3+\nu)(1-\nu)\sinh\alpha_m \cosh\alpha_m - (1-\nu)^2\alpha_m + 2m\pi\lambda\cosh^2\alpha_m}$$

All terms in the above expressions are defined in the main body of this article except α_m and λ which are given as:

$$\alpha_m = \frac{m\pi b}{2a}$$

and

$$\lambda = \frac{EI}{aD}$$

The term λ represents the flexural rigidity of any support beam divided by the flexural rigidity of the plate. For the simply supported configuration considered in the article this term is zero. The curvatures of the plate are given by the following equations:

$$\kappa_x = \frac{\partial^2 w}{\partial x^2}$$

$$\kappa_y = \frac{\partial^2 w}{\partial y^2}$$

$$\kappa_{xy} = \frac{\partial^2 w}{\partial x \partial y}$$

The curvatures convert to moments through the appropriate constitutive relations:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1+\nu) \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

The moments are thus determined from the curvatures which involve the second spatial derivatives of the displacement field. The following FORTRAN subroutine was created to evaluate the transverse displacement and moments for the plate configuration considered in this article and with arbitrary aspect ratio.

```
SUBROUTINE LEVY_SOLUTION(MMAX,X,Y,A,B,Q,E,NU,T,W,MX,MY,MXY,M1,M2,MV)

IMPLICIT NONE

REAL(8),INTENT(IN)::X,Y,A,B,Q,E,NU,T
REAL(8),INTENT(INOUT)::W,MX,MY,MXY,M1,M2,MV

REAL(8)::D
INTEGER::M
INTEGER::MMAX
REAL(8)::ALPHAM,AM,BM,DENOMINATOR
REAL(8)::KXX,KYY,KXY
REAL(8),PARAMETER::LAMBDA=0D0
REAL(8)::TERM
REAL(8),PARAMETER::PI=ACOS(-1D0)
REAL(8)::W_TIMOSHENKO

D=E*T**3/(12D0*(1D0-NU**2))

MX=0D0 ; MY=0D0 ; MXY=0D0 ; W=0D0

DO M=1,MMAX,2

          ALPHAM=M*PI*B/(2D0*A)

          AM=NU*(1D0+NU)*DSINH(ALPHAM)-NU*(1D0-NU)*ALPHAM*DCOSH(ALPHAM)-M*PI*LAMBDA*(2D0*DCOSH(ALPHAM)+ALPHAM*DSINH(ALPHAM))
          BM=NU*(1D0-NU)*DSINH(ALPHAM)+M*PI*LAMBDA*DCOSH(ALPHAM)

          DENOMINATOR=(3D0+NU)*(1D0-NU)*DSINH(ALPHAM)*DCOSH(ALPHAM)-(1D0-NU)**2*ALPHAM+2D0*M*PI*LAMBDA*(DCOSH(ALPHAM))**2

          TERM=(4D0/M**5/PI**5)

          AM=AM*TERM/DENOMINATOR
          BM=BM*TERM/DENOMINATOR

          W=W+Q*A**4/D*((4D0/PI**5/M**5+AM*DCOSH(M*PI*Y/A)+BM*M*PI*Y/A*DSINH(M*PI*Y/A))*DSIN(M*PI*X/A))

          KXX=Q*A**4/D*(-
          ((4/(PI**5*M**5)+AM*DCOSH((M*PI*Y)/A)+(DSINH((M*PI*Y)/A)*BM*M*PI*Y)/A)*DSIN((M*PI*X)/A)*(M*PI)**2)/A)/A

          KYY=Q*A**4/D*(DSIN((M*PI*X)/A)*(AM*DCOSH((M*PI*Y)/A)*(M*PI)**2+BM*M*PI*DCOSH((M*PI*Y)/A)*M*PI+(BM*M*PI*Y*DSINH((M*P
          I*Y)/A)*(M*PI)**2)/A+BM*M*PI*DCOSH((M*PI*Y)/A)*M*PI))/A**2

          KXY=Q*A**4/D*(DCOS((M*PI*X)/A)*M*PI*(AM*DSINH((M*PI*Y)/A)*M*PI+DSINH((M*PI*Y)/A)*BM*M*PI+(BM*M*PI*Y*DCOSH((M*PI*Y)/
          A)*M*PI)/A))/A**2

          MX=MX+D*(KXX+NU*KYY)
          MY=MY+D*(NU*KXX+KYY)
          MXY=MXY+D*(1D0+NU)*KXY

ENDDO

END SUBROUTINE
```

This subroutine was used to determine some of the results presented in this article and the results were verified independently using the Mathematica software – see acknowledgements.