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Navier Solution for the Simply Supported Rectangular Plate under UDL

The reason for writing this technical note was to examine how a theoretical exact Kirchhoff solution cast as an infinite series converges as the number of terms in the series is increased. The Navier plate solution is chosen and it is of interest to see how many terms are required to produce solutions of reasonable engineering accuracy – say 1%. Of interest also is the nature of the approximation, i.e., how does the error in the solution manifest itself?

Navier Solution

The Navier solution for the rectangular plate simply supported on all sides and under a uniformly distributed load, q_o, as shown in Figure 1, is presented in Chapter 5 of Timoshenko's text.

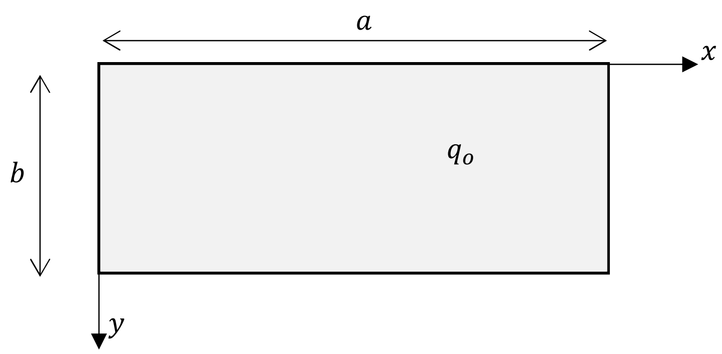


Figure 1: Rectangular plate

The displacement field, w, is presented as a double infinite summation of sine terms as shown in Eq. (1), where D is the flexural rigidity of the plate.

$$w = \frac{16q_o}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{mn\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \qquad m = 1, 3, 5, \ldots \text{ and } n = 1, 3, 5, \ldots \tag{1}$$

This solution for the displacement satisfies the Germain-Lagrange equation, Eq. (2), for thin plates based on the Kirchhoff hypothesis so that all internal static and kinematic conditions are satisfied with the appropriate linear elastic constitutive relations between curvatures and moments.

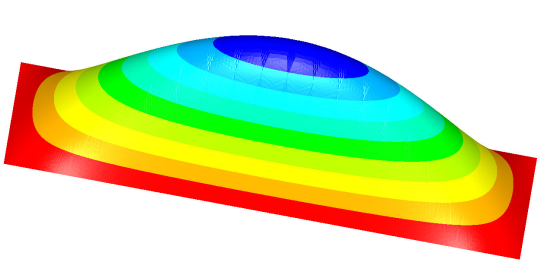
$$\nabla^4 w = \frac{q(x,y)}{D} \tag{2}$$

The displacement field satisfies the kinematic boundary condition of zero transverse displacement around the boundary, and the moment fields derived from this displacement field satisfy the static boundary condition of zero normal moment on the boundary.

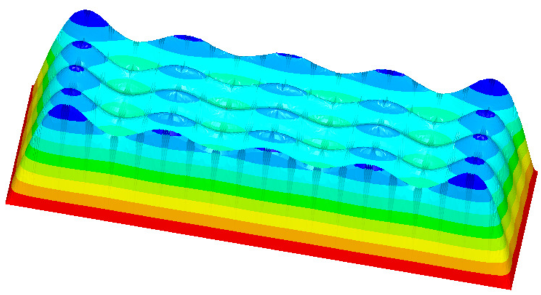
Each term in the summation of Eq. (1) thus satisfies all the relevant conditions for an exact elastic solution. But the solution only tends to the exact one as the indices m and n tend to infinity. The reason for this is that the load on the plate, $q(x,y)$ for a particular truncation of the infinite series is not uniform and its resultant force is not equal to that of the applied load q_o. The load on the plate is given in Eq. (3).

$$q = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{16q_o}{\pi^2 mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad m = 1,3,5,\dots \text{ and } n = 1,3,5,\dots \qquad (3)$$

Consider a plate of dimensions $a = 2m$ and $b = 0.6m$ and with a UDL of $q_o = 1000Pa$. The resultant load is then $Q_o = 1200N$. We will show solutions for truncated series with equal upper values of the two indices such that $m = n = p$. The range of the common index p will be $1 \leq p \leq p_{max}$ with $p = 1,3,5,\dots$ The load distributions for p_{max} equal to 1 and 10 are shown in Figure 2.



$p_{max} = 1\ ;\ q_{max} \cong 1621Pa\ ;\ Q \cong 788N$



$p_{max} = 9\ ;\ q_{max} \cong 1384Pa\ ;\ Q \cong 1105N$

Figure 2: Load distributions for p_{max} equal to 1 and 10

The load distribution for $p_{max} = 1$ shows a significant (sinusoidal) variation over the area of the plate with a peak pressure of 1621Pa at the centre of the plate, i.e., some 60% greater than the actual value. The resultant force, at 788N, is just under 70% of the correct value of 1200N. The distribution is improved by increasing p_{max} but even for $p_{max} = 10$ the distribution is still 'lumpy' with a resultant force just over 90% of the correct value. The way in which the resultant force converges with p_{max} is shown in Figure 3 where the relative error, e, for a given quantity, r is defined as in Eq. (4).

$$e = \frac{r - \tilde{r}}{r} \qquad (4)$$

where r is the exact value or a surrogate for the exact value and $\tilde{r}$ is the approximate value from the truncated infinite series.

The convergence of the transverse deflection and the principal moments at the centre of the plate ($x = a/2\ ;\ y = b/2$), where they are maximum, is also shown in Figure 3. The relative errors were calculated using the known value of the total load and deflections and moments taken for $p_{max} = 249$ – this being deemed sufficiently large so as to make no observable difference in the convergence plots.

In the linear-linear plot which includes the sign of the relative error, we observe that whilst the convergence of the total load is monotonic, that of the deflection and moments is oscillatory. In the log-log plot which uses the absolute value of the relative error, it is seen that a value for p_{max} of nearly 99 is required to reduce the error in the total load below 1%. The rates of convergence of the quantities plotted in Figure 3 are indicated by the slopes of the curves as they become linear towards the right of the figure, i.e., beyond the pre-asymptotic region at the left of the figure. The rate of convergence for the displacement is greater than that for the moments which is greater than that for the load. The number of terms in the series required to achieve a given accuracy is, of course, dependent on the convergence rate but also on the starting value, i.e., the value of the quantity for $p_{max} = 1$. For the plate being considered, the deflection and moments are seen to converge to values with errors around 1% for $p_{max} \cong 11$.

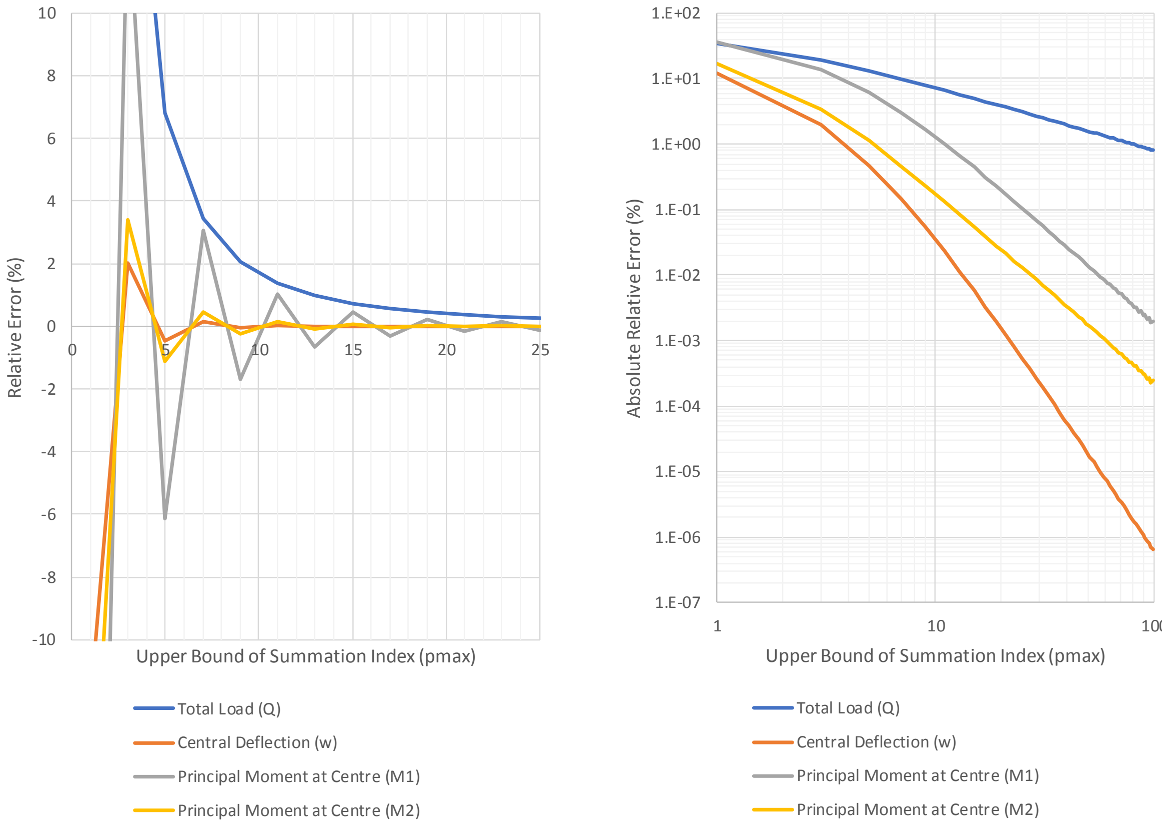


Figure 3: Convergence of quantities of interest

Closure

It might be seen as serendipitous that a crude representation of the load distribution and resultant force, as seen with low values of p_{max} can lead to reasonably accurate predictions of the central deflections and moments. However, it is seen that the for $p_{max} = 1$ whilst the resultant force is significantly lower than the exact value, the distribution is biased towards the centre of the plate. The addition of further terms in the series increase the total load and also flatten the distribution. Clearly, though not investigated here, the error in these quantities will vary over the plate. In terms of the central deflection, Timoshenko notes that for a square plate the error obtained for $p_{max} = 1$ is only about 2.5% - for the rectangular plate considered in this technical note, though, the error is nearer to 12%.

From a practical perspective, the oscillatory nature of the convergence is potentially rather useful in that any two adjacent solutions, e.g., for $p_{max} = 3$ and $p_{max} = 5$ offer the engineer upper and lower bounds on the exact solution. Further, if the engineer wishes to use the Navier solution to provide a conservative prediction of the deflection and moments then he or she might choose to use upper bound predictions and for the definition of the relative error, Eq. (4), this would mean taking values with a negative relative error, i.e., $p_{max} = 1, 5, 9, \ldots$ or $p_{max} = 4s - 3$ $(s = 1, 2, 3, \ldots)$.