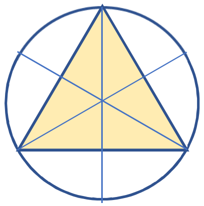
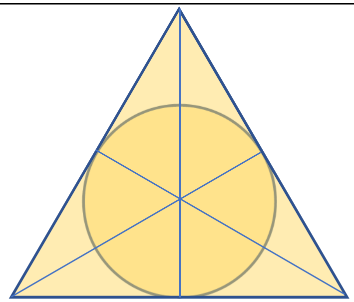
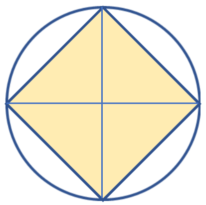
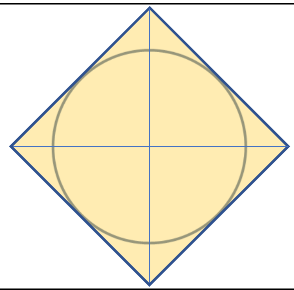
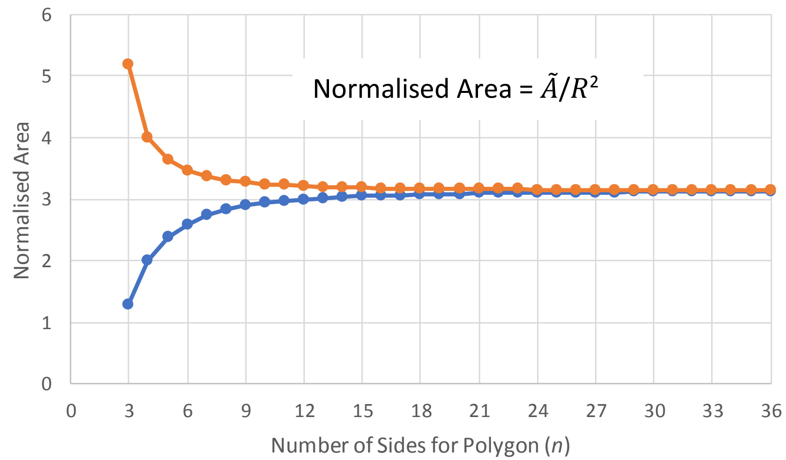
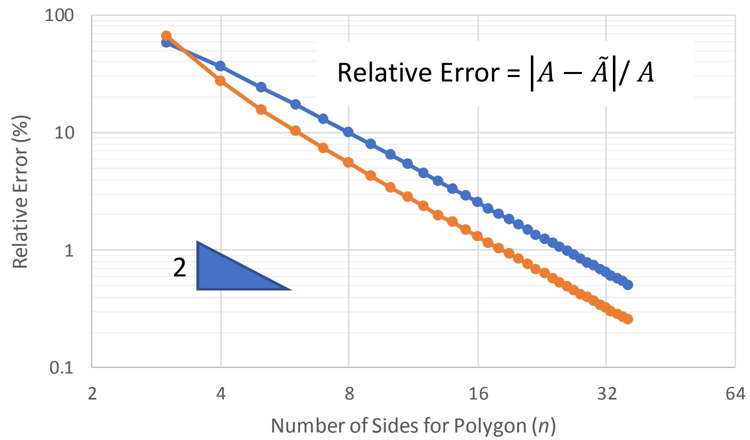
Convergence of Numerical Solutions (Basic Example)

The area of a circle, A, or radius, R, may be approximated by the area of regular polygons, $\tilde{A}$, of *n* sides that inscribe (sit inside) or circumscribe (sit outside) the circle. As the number of sides, *n*, increases the approximations get progressively closer to the exact solution. Convergence for the inscribed polygon is from below the exact solution whereas for the circumscribed polygon it is from above the exact value so that for a given value of *n*, lower and upper bounds to the exact solution can be obtained.

$$n \cos\theta \sin\theta \le A \le n \cot\theta \quad \text{where} \quad \theta = \frac{\pi(n-2)}{2n}$$

n	Inscribed	Circumscribed
3		
4		



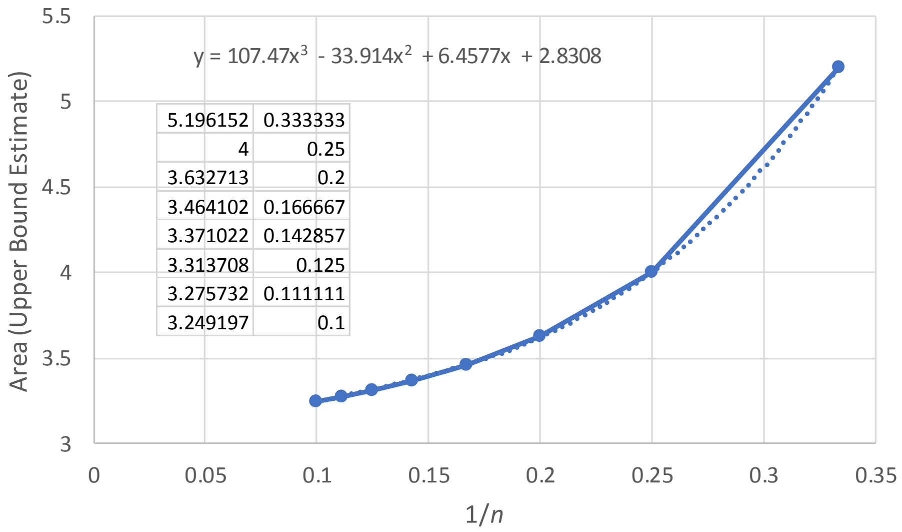


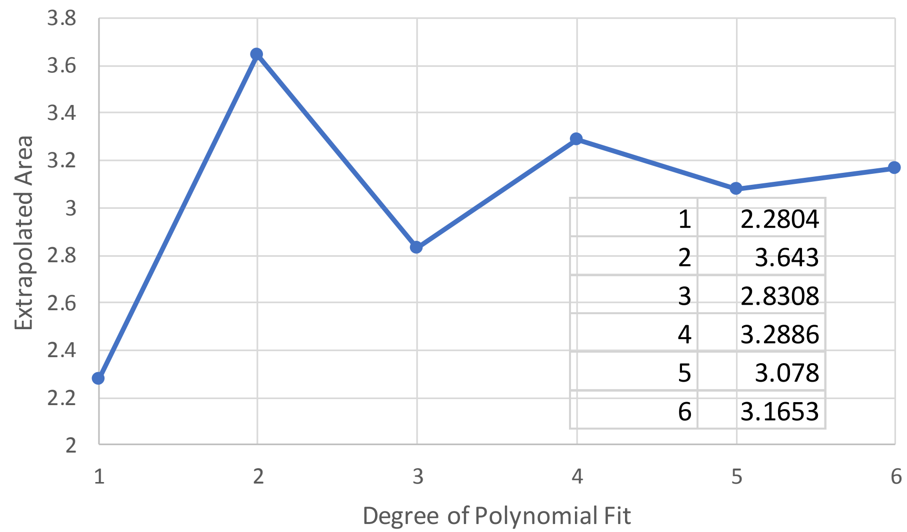
The normal convergence plot shows upper and lower bound solutions converging in an asymptotic manner to the theoretical value of the normalised area (π). The log-log plot, which requires knowledge of the exact solution, shows both solutions converging, after the pre-asymptotic region, to the exact solution with a rate of convergence of two. In order to achieve a solution with less than 1% error requires 19 and 26 sides respectively for the circumscribed and inscribed approximations.

A Counter-Example to a Published Approach to Estimating the Mesh Size

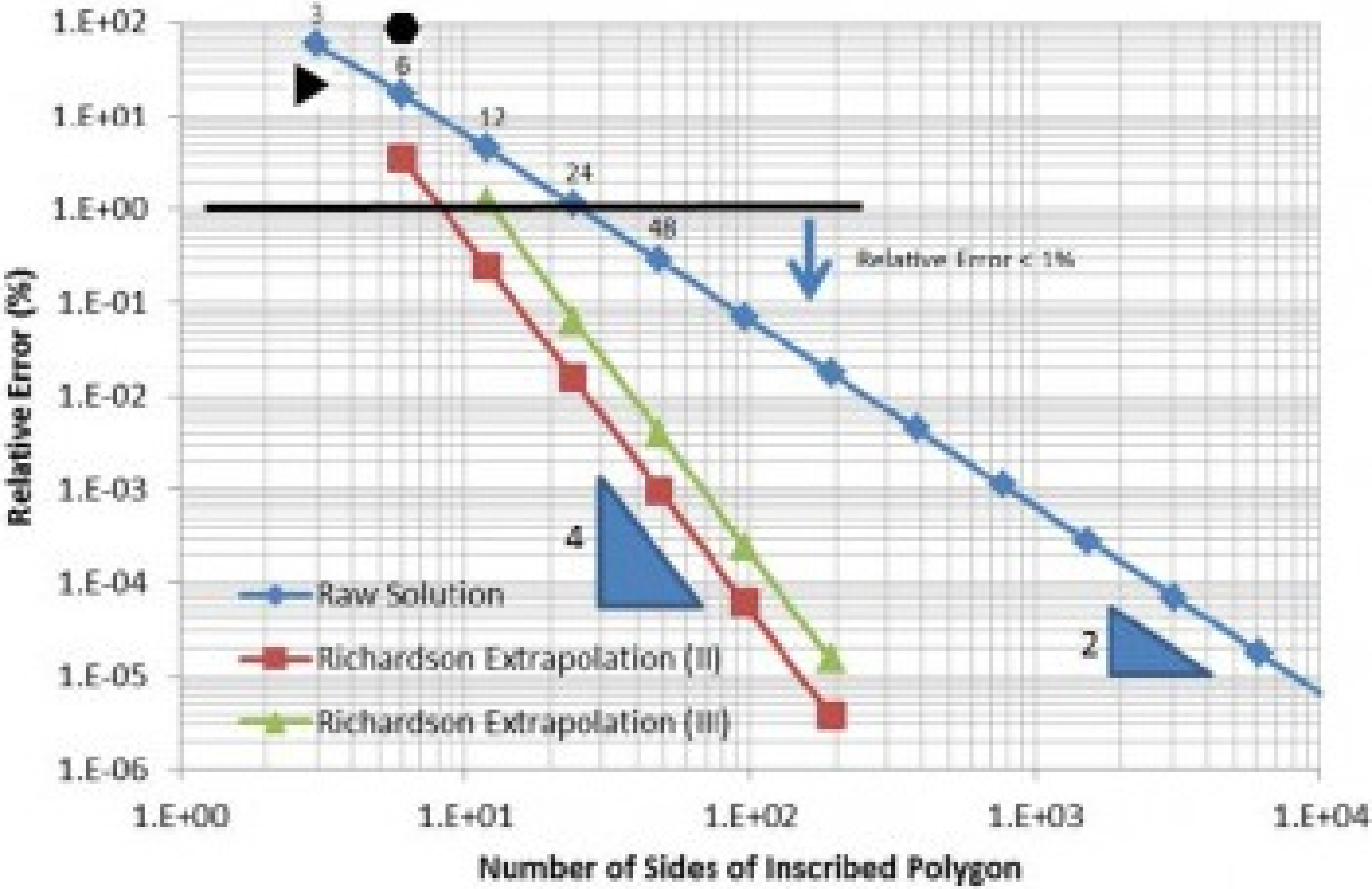
An interesting technique for extrapolating to the exact solution from a series of approximate finite element solutions was presented in https://enterfea.com/correct-mesh-size-quick-guide/. In essence it was suggested by linearly extrapolating a plot of FE response versus the inverse of the number of nodes one could obtain a good estimate of the theoretical solution.

Using the first eight results for the upper bound approximation of the area (see example above) Excel has been used to fit polynomials of degree 1 (linear) to six to the results. The value of polynomial when 1/*n* = zero has then been plotted.





The exact value of the area is π and whilst with increasing degree of polynomial fit the exact value is approached, the value obtained using the linear polynomial is extremely poor. Thus, the proposed method is not a good one! The correct approach to extrapolation would be to use Richardson Extrapolation. The following figure shows Richardson Extrapolation applied to the lower bound convergence.



1.E+02
1.E+01
1.E+00
1.E-01
1.E-02
1.E-03
1.E-04
1.E-05
1.E-06
Relative Error (%)
3
6
12
24
48
Relative Error < 1%
4
2
Raw Solution
Richardson Extrapolation (II)
Richardson Extrapolation (III)
1.E+00
1.E+01
1.E+02
1.E+03
1.E+04
Number of Sides of Inscribed Polygon