Performance of CFEs on Problems Governed by the Lamé Equations

Finite elements found in most commercial finite element systems are normally based on the conforming finite element (CFE) formulation. Elements based on this formulation adopt shape functions that define how the element can displace and also, if they are isoparametric, the shape of the element. These shape functions are normally low degree polynomials, e.g., linear or quadratic. The shape functions interpolate nodal values of displacement which are normally the unknowns in an FE problem. Strains determined by differentiating these displacements are by definition compatible and stresses determined from these strains using the appropriate constitutive relations then satisfy two of the three conditions necessary for an elastic solution. The third condition, namely equilibrium, is not normally satisfied exactly but, rather, the FE solution attempts to minimise the equilibrium defaults present in the solution to provide an approximate solution that is, hopefully, close to the exact solution.

In testing FE software and, indeed, in selecting an appropriate element for a given problem, one may select to run a problem for which there is a closed-form solution. In this manner one can confirm the coding of the software and observe the level of mesh refinement required to capture the solution to sufficient engineering accuracy.

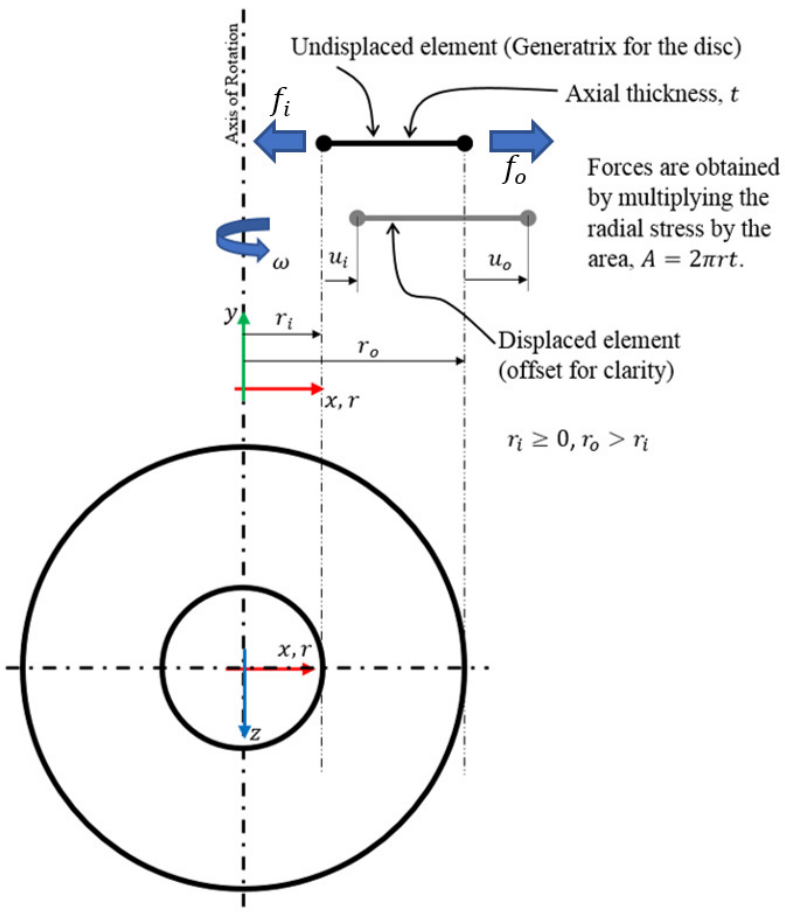

Figure 1: A Lame Finite Element

This technical note examines the performance of the CFE axisymmetric membrane element, shown in Figure 1, for closed-form solutions governed by the Lamé equations. The Lamé equations, shown in Eq. (1) for an isotopic material with plane stress constitutive relations, express the quantities of engineering interest, radial and hoop stress, σ_r and σ_h respectively, and radial displacement, u, in terms of the two Lamé coefficients, a and b which are determined from the boundary conditions for the problem. The equations are simplified by excluding body and thermal loading terms which are not considered in this technical note.

$$\sigma_r = a - \frac{b}{r^2} \qquad \text{Radial Stress} \qquad (1a)$$

$$\sigma_h = a + \frac{b}{r^2} \qquad \text{Hoop Stress} \qquad (1b)$$

$$u = \frac{r}{E}(\sigma_h - \nu\sigma_r) \qquad \text{Radial Displacement} \qquad (1c)$$

The first two equations, Eq. (1a) and Eq. (1b) are statements of equilibrium of internal stresses and the third equation, Eq. (1c) is a statement of compatibility between radial strains and radial displacements.

For the thick cylinder shown in Figure 2, two loading cases will be considered conforming to the Lamé coefficients a=100kPa, b=0 and a=0, b=100kN. These have been chosen with a view to evaluating the performance of the corresponding displacement element found in commercial FE systems.

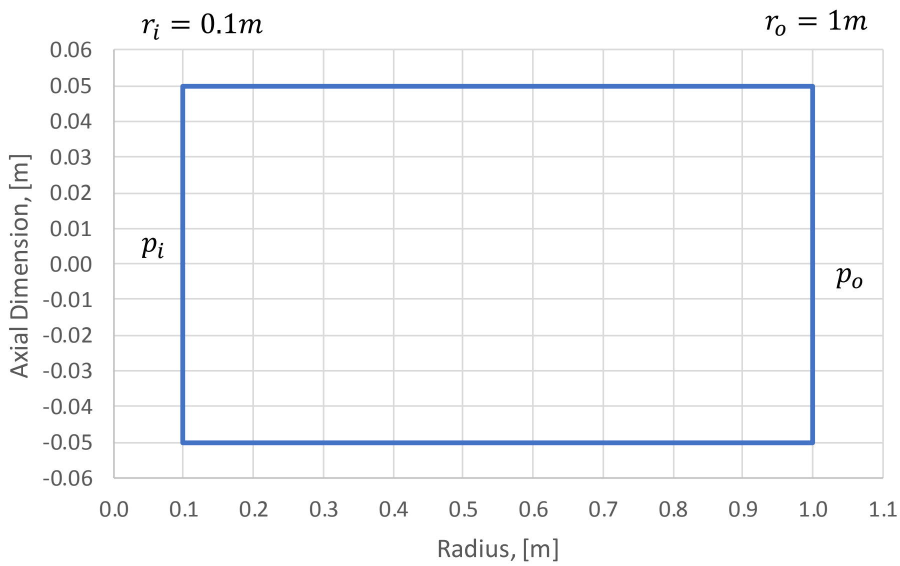

Figure 2: Thick cylinder under internal and external pressures

The nodal forces for specified values of the Lamé coefficients are obtained from Eq. (1) as shown in Table 1.

Case	a, [kPa]	b, [kN]	f_i, [kN]	f_o, [kN]	p_i, [kPa]	p_o, [kPa]
1	100	0	-2π	$+20\pi$	-100	100
2	0	100	$+200\pi$	-20π	10000	-100

Table 1: Load cases for thick cylinder example

The solution to the two load cases in terms of the radial and hoop stresses are shown in Figure 3.

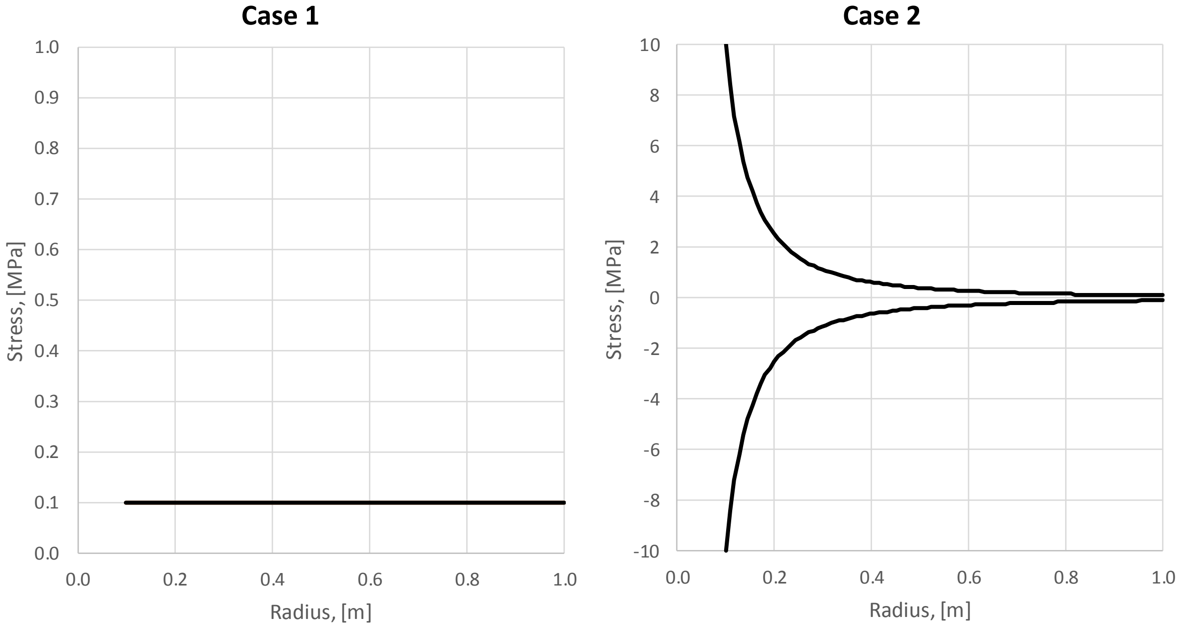

Figure 3: Radial and hoop elastic stresses for thick cylinder example

Commercial FE systems typically offer axisymmetric shell elements in lower-order (two nodes) and higher-order (three nodes) forms and often provide an option to switch off the bending part of the shell leaving an axisymmetric membrane capability.

For the first load case considered for the thick cylinder, the CFEs recover the exact solution with a single element. This is not surprising since the stresses are constant through the wall of the cylinder. For the second load case, however, the stress recovery is rather poor as illustrated in Figure 4 where the percentage error in the stress (radial and hoop stresses being equal both in the theoretical solution and the FE solution) at the inner radius has been plotted against number of elements.

For both lower and higher order elements a significant number of elements are required before the asymptotic region of uniform convergence rate is reached. Both elements underpredict the true stress and by significant amounts for coarse meshes and this could lead to a potentially unsafe design were the engineer not to detect the poor quality of the result.

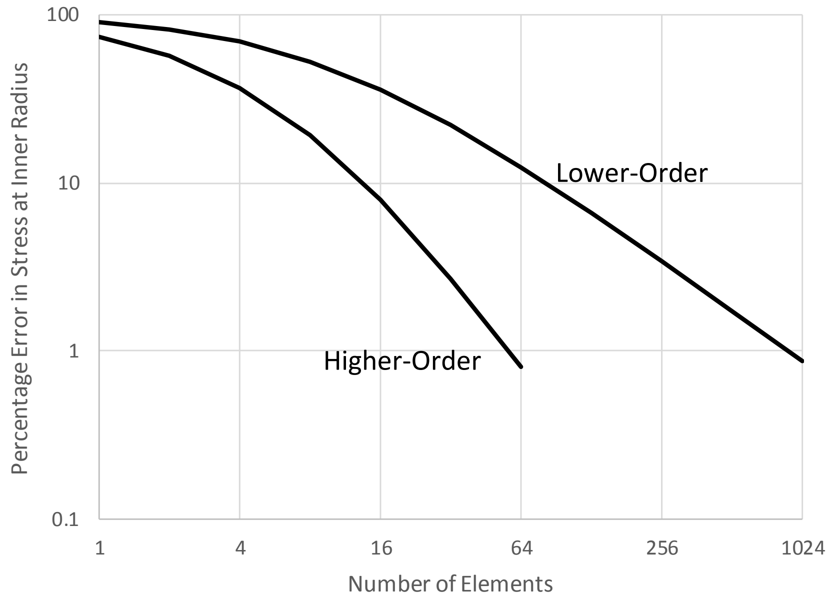

Figure 4: Convergence of elastic stress for load case 2

In order to establish elastic limit loads one needs to choose an appropriate failure criterion. Assuming the material used for the cylinder to be ductile then the von Mises criterion is most appropriate and this has been used together with an elastic, perfectly-plastic material model to produce the results shown in Figure 5.

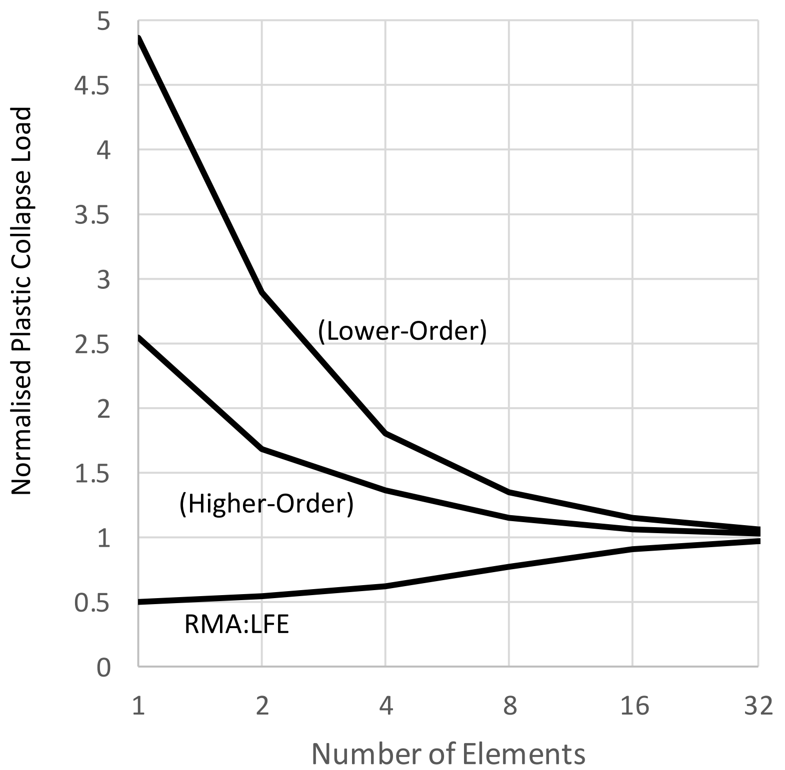

Figure 5: Convergence of plastic limit load for load case 2 (von Mises yield criterion)

The conventional conforming elements produce results that, as in the elastic case, converge unsafely from above. The results from a Lame Finite Element (LFE) are also shown which, being of an equilibrium finite element (EFE) formulation, converge safely from below the true collapse load.