Stress Fields in the Standard Four-Noded Conforming Finite Element

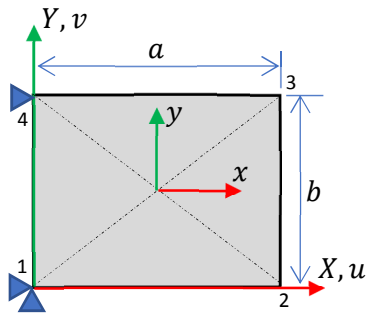
Introduction

In considering the capabilities of a finite element, practicing engineers tend to think in terms of the element's ability to model stresses rather than displacements. For example, if the engineer needs to model a component subject to an applied bending moment then he/she would look for an element capable of modelling a constant moment stress field. For standard conforming finite elements (CFE) the engineer will know the element's capabilities in terms of displacements since this is given directly by the shape functions, e.g., the four-noded plane stress element has bilinear shape functions which, for isoparametric elements, describe both the shape of the element and also the displacement fields.

The element's displacement fields can be transformed into strain and then stress fields and doing so is instructive in terms of understanding the element's capabilities. This article considers the four-noded plane stress element presenting and discussing the five independent modes of stress that it is capable of modelling.

Stresses in Terms of Nodal Displacements

The four-noded element considered in this article is shown in Figure 1. We will only consider square or rectangular elements here as shown in Figure 1.



$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1-\nu^2 & 0 & 0 & y & \nu x\\ 0 & 1-\nu^2 & 0 & \nu y & x\\ 0 & 0 & c & cx & cy\end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\end{Bmatrix}, where\ c = \frac{1-\nu}{2}$$

$$\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\end{Bmatrix} = \frac{1}{ab}\begin{bmatrix}b & 0 & 0 & 0 & -a\nu\\ -b\nu & 0 & 0 & 0 & a\\ 0 & b & 0 & 0 & 0\\ -1 & 0 & 1 & 0 & a\\ 0 & -1 & 0 & 1 & -1\end{bmatrix}\begin{Bmatrix}u_2\\ v_2\\ u_3\\ v_3\\ v_4\end{Bmatrix}$$

$$\begin{bmatrix}\frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y}\\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}0 & 0 & 0 & 0 & \nu+c\\ 0 & 0 & 0 & \nu+c & 0\end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\end{Bmatrix}$$

Figure 1: Four-noded element

The derivation of the stress fields for the element is shown in the appendix to this technical note. The key point to note about the element stress field is that there are five independent modes of stress, corresponding to the five independent modes of displacement – originally eight modes but minus the three rigid-body modes of displacement that cause zero stress. As these modes of stress were derived from modes of displacement they are, by definition, kinematically admissible, i.e., there are corresponding compatible strain fields. The test for static admissibility is whether or not the stress fields are in equilibrium with zero body forces. It is seen in the third equation in Figure 1, that whilst the first three (constant) stress fields are statically admissible, the fourth and fifth fields have non-zero body forces and are therefore not statically admissible with a problem involving no body force loading. Furthermore, as the body forces are a function of Poisson's ratio, it is unlikely that any engineering problem of interest will have body loads that conform with these body forces.

TO BE CONTINUED!