Ramsay Maunder ASSOCIATES *Finite Element Specialists and Engineering Consultants*

# Research and Development at Ramsay Maunder Associates

Seminar for Final Year Civil Engineering Students, University of Surrey (9th November 2016)

Angus Ramsay PhD, FIMechE & Edward Maunder PhD, FIStructE

**Keywords**: Simulation Governance, Verification & Validation, Reinforced Concrete Slabs, Steel Plates, Elastic Solutions, Limit Analysis, Yield Line Techniques, Lower Bound Techniques, Equilibrium Finite Elements.

## Introduction

This seminar was arranged following correspondence between Angus Ramsay and Juan Sagaseta of the University of Surrey. Juan had seen and solved, correctly, the 'And Finally …' question posed in the August edition of the IStructE's 'The Structural Engineer'.

Ramsay Maunder Associates (RMA) was established as a partnership in 2004 and incorporated in 2009. The aim of the company was/is the commercial exploitation of Equilibrium Finite Elements (EFE) as a useful addition to the practising engineers' tool kit. The work has been self-funded, through undertaking commercial contracts as specialists in finite element analysis.

During the years a software tool (EFE) has been developed for, amongst others, the lower bound limit analysis of reinforced concrete (RC) slabs and steel plates. RMA are working in collaboration with LimitState (a spinout company from the University of Sheffield) with a view to incorporating EFE, for RC slabs, into their existing upper bound (yield line) software (SLAB).

Over the last 18 months, a significant effort has been involved in writing articles for an audience of practising engineers, illustrating the virtues of the EFE method both for RC and for steel:

- The Structural Engineer (IStructE)
- Structure Magazine (US)
- Concrete Magazine (UK)
- Engineering & Technology Reference (Institution of Engineering & Technology)
- Benchmark Magazine (NAFEMS)
- IABSE, SED on Lower Bound Limit Analysis of Steel Plates (in preparation)

The presenters are, variously, involved as:

- Member of NAFEMS Education & Training Working Group
- Independent Technical Editor for the NAFEMS Benchmark Challenge
- Professional Simulation Engineer (PSE) Assessor for NAFEMS
- Academic Qualifications Panel for the IStructE

The publication of such articles and the involvement with such bodies enable RMA to engage an audience of practising engineers to explain the virtues of EFE.

The work RMA have undertaken, particularly on the limit analysis of steel plates, has provided some new and interesting insights into how steel plates collapse. In particular, it has been demonstrated that the current practice of assessing such structural members using the traditional yield line technique is flawed. The use of EFE has also enabled the establishment of more accurate collapse loads, which are often significantly greater than published values thereby allowing the practising engineer an opportunity to squeeze more strength out of his/her structure and thus reduce the use of structural steel.

## Simulation Governance (Verification & Validation)

Simulation governance is the process of achieving a good match between numerical simulation, e.g., finite element analysis, and physical reality. It can only occur when the engineer understands the mathematical model that governs physical reality and when the numerical simulation process provides results that compare well with the mathematical model. The process that provide the foundations for the three columns are known as verification and validation as illustrated in Figure 1.

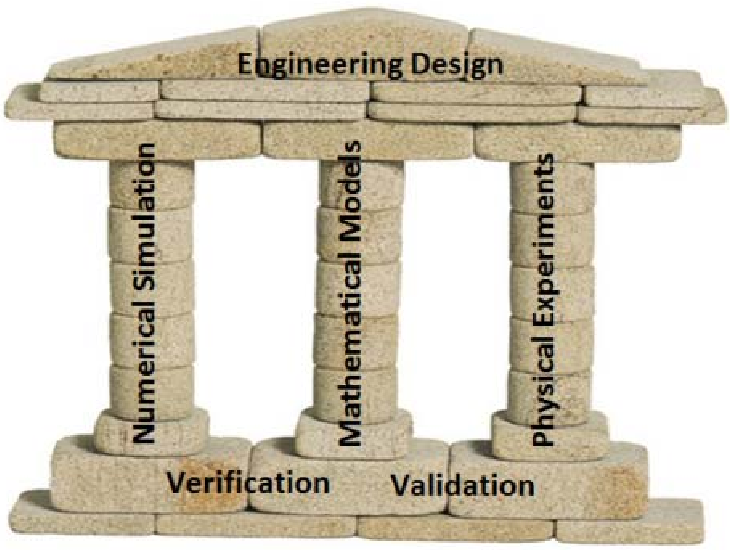



**Figure 1**: Simulation governance and Verification & Validation

A typical strength of materials solution to a problem in linear elasticity is one having a known theoretical solution and the mathematical model requires the following three conditions to be satisfied everywhere, i.e., in a pointwise sense.

**Table 1**: Conditions for a theoretically exact solution

| Statics | Constitutive | Kinematics |
|---|---|---|
| **Equilibrium** between the internal stresses, the body forces and the static boundary conditions | Stress/Strain (Hooke's Law) | **Compatibility** between the strains and the displacements and enforcement of the kinematic boundary conditions |

Strength of materials solutions provide a useful library of *known* theoretically exact solutions that the engineer may draw upon for the design/assessment of many simple components and structures. However, as soon as the structure falls outside the scope of this library, e.g., more complex geometry, material properties, boundary or loading conditions, then the engineer is faced with an *unknown* theoretical solution. Clearly, a theoretical solution does still exist but it is not known and it is the engineer's task, if his/her design is to be a success, to find a good approximation to the theoretically exact solution.

This is where numerical simulation and, in particular, the finite element method steps up to assist the engineer. The finite element method is approximate however, with sufficient mesh refinement the approximate solution should converge to the theoretical solution. Let us examine the nature of the approximations present in two 'pure' finite element formulations.

**Table 2**: Conditions satisfied weakly or strongly for 'pure' finite element formulations

| | **Statics** | **Constitutive** | **Kinematics** |
|---|---|---|---|
| **Strength of Materials** | Strong | Strong | Strong |
| **Conforming (CFE)** | Weak | Strong | Strong |
| **Equilibrium (EFE)** | Strong | Strong | Weak |

To the practising engineer, concerned with ensuring the strength of his/her structure of all the above conditions it is the satisfaction of equilibrium in a strong sense that is paramount. Reinforcement, for example, can be placed to withstand the moments from a finite element analysis BUT if these moments are not in equilibrium with the applied load then there could be an issue further down the line. Edward L. Wilson, original developer of the SAP finite element software, expressed this idea quite succinctly in his book Three Dimensional Static and Dynamic Analysis of Structures:

'Equilibrium is Essential – Compatibility is Optional'

http://www.edwilson.org/book/02-equi.pdf

This is all very well but as indicated above, CFEs (the element formulation used in most if not all commercial FE software) only satisfies equilibrium in a weak sense! The redeeming feature though is that with mesh refinement weak equilibrium is strengthened so that, in the limit, one cannot distinguish it from strong equilibrium. This does though require the engineer to be aware of this point and actually to undertake mesh refinement and convergence studies! It should be noted in the context of equilibrium in the CFE formulation that nodal forces around each element do provide a set of point forces that hold the element in equilibrium.

There is, though, an alternative formulation, known as the Equilibrium Finite Element Formulation, which always provides strong equilibrium irrespective of the level of mesh refinement. EFE's are, thus, perfect for the structural engineer! This is not a new formulation but developments over the last thirty or so years have ironed out many of the issues faced by engineers in the early days when attempting to implement this method. Indeed a book on Equilibrium Finite Element Formulations will soon be published detailing some of the techniques used in the software EFE – see Figure 2.

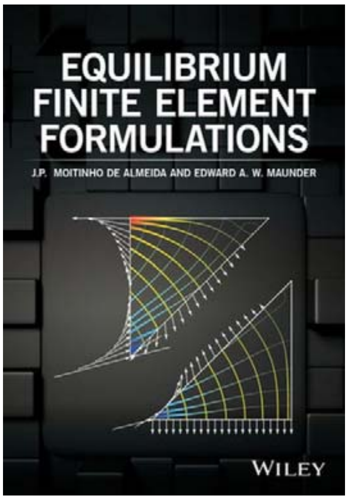
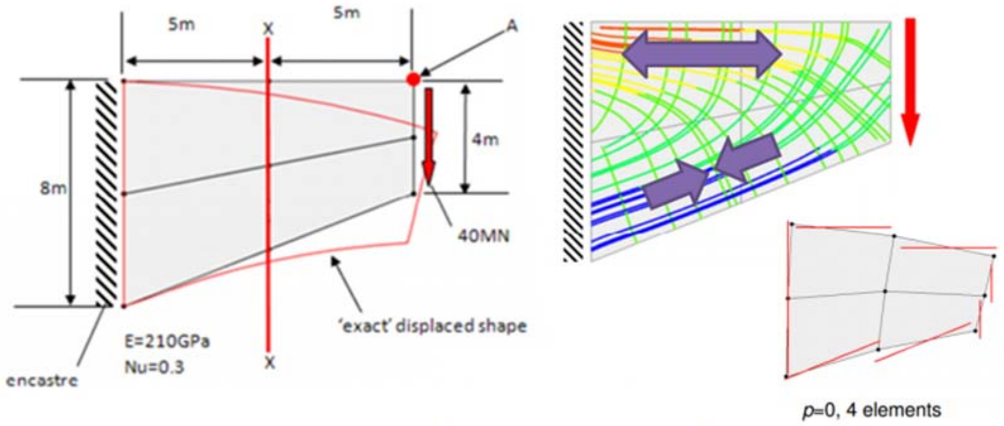



**Figure 2**: Equilibrium finite element formulations (to be published in March 2017)

The differences between CFE and EFE approximations can be illustrated with the simple tapered cantilever example shown in Figure 3. A rather coarse mesh of 2x2=4 elements has been used and the engineer is interested in establishing, from the finite element stress fields, the stress resultants along the section XX; these can of course be easily calculated by hand by appeal to static equilibrium conditions.




| | Displacement (mm) | Normal (MN) | | Tangential (MN) | | Moment (MNm) | |
|---|---|---|---|---|---|---|---|
| | | Left | Right | Left | Right | Left | Right |
| EFE (*p*=1) | -3.58 | 0 | 0 | 40 | 40 | 200 | 200 |
| CFE (four-noded) | -2.56 | 27.1 | 19.7 | 14.5 | 80.0 | 200.2 | 62.8 |
| CFE (eight-noded) | -3.48 | 0.0 | 0.0 | 39.5 | 38.9 | 205.2 | 220.7 |
| 'Exact' | -3.54 | 0 | 0 | 40 | 40 | 200 | 200 |

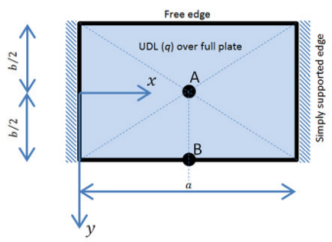
**Figure 3**: Tapered cantilever example

The finite element stresses (for both CFE and EFE models) are generally not continuous across element boundaries and so the stress resultants have been calculated and presented for both sides of the section. The first thing to note is that the EFE model recovers the exact stress resultants on both sides of the section. This is as to be expected because the formulation provides strong equilibrium. In contrast, the CFE results are not exact and are different on the two sides of the section. The results for both four and eight noded CFEs have been presented and even with the higher order (eight noded) element, there is still a 10% error between the CFE result and the exact value for the moment resultant on the right of the section!

## Assessment of a Steel Plate

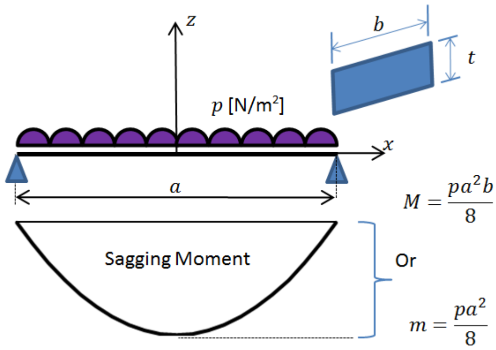
If good simulation governance is to be observed then it is important that the correct mathematical model be adopted for the structural member under consideration, c.f., Figure 1. If an incorrect mathematical model is adopted then the process of validation will fail.

As an example of the use of an incorrect mathematical model a uniformly loaded rectangular plate (steel), simply supported on two opposite sides will be considered. Whilst, if the width (*b*) of the plate is small in comparison to its span, a beam representation might be appropriate, does this remain the case as the width increases in relation to the span? Beam and plate representation of this plate configuration are shown in Figure 4.




| | Point A | | | Point B | |
|---|---|---|---|---|---|
| | $x = a/2 ; y = 0$ | | | $x = a/2 ; y = \pm b/2$ | |
| $b/a$ | $w = \alpha_1 \frac{qa^4}{D}$ | $M_x = \beta_1 qa^2$ | $M_y = \beta_1' qa^2$ | $w = \alpha_2 \frac{qa^4}{D}$ | $M_x = \beta_2 qa^2$ |
| | $\alpha_1$ | $\beta_1$ | $\beta_1'$ | $\alpha_2$ | $\beta_2$ |
| 0.5 | 0.01377 | 0.1235 | 0.0102 | 0.01443 | 0.1259 |
| 1.0 | 0.01309 | 0.1225 | 0.0271 | 0.01509 | 0.1318 |
| 2.0 | 0.01289 | 0.1235 | 0.0364 | 0.01521 | 0.1329 |
| ∞ | 0.01302 | 0.1250 | 0.0375 | 0.01522 | 0.1330 |

(a) Plate representation




(b) Beam representation

The bending moment $M$ for the beam has the usual units Nm. However, when considering plates it is more usual to use moments per unit width, $m$, which then has units Nm/m.

**Figure 4**: Two representations of a plate problem

Let us consider a wide plate with an aspect ratio of 2 where a=1m, b=2m, t=0.01m and the material has a yield stress of $S_y$=275MPa firstly using a beam representation and then a plate representation.

### Mathematical Model - Beam

The pressure to cause first yield and plastic collapse (through the development of a central sagging yield line) can be established once the maximum moment in the beam is determined simply by equating this moment to the moment required to cause first yield and plastic collapse:

$$m_f = \frac{S_y t^2}{6} \quad \text{Moment to cause first yield of section} \quad (1)$$

$$m_c = \frac{S_y t^2}{4} \quad \text{Moment to cause collapse of section} \quad (2)$$

The pressures are, respectively, 36.7kPa and 1.5x36.7=55kPa. The Steel Construction Institute offers practising engineers advice on the design of structural steel members in its Steel Designers' Manual, [1]. An extract from the SDM relevant to the plate being considered is shown in Figure 5.

Ultimate load capacity (kN/m$^2$) for floor plates simply supported on two edges stressed to 275 N/mm$^2$

| Thickness on plain mm | Span (mm) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 600 | 800 | 1000 | 1200 | 1400 | 1600 | 1800 | 2000 |
| 4.5 | 20.48 | 11.62 | 7.45 | 5.17 | 3.80 | 2.95 | 2.28 | 1.87 |
| 6.0 | 36.77 | 20.68 | 13.28 | 9.20 | 6.73 | 5.20 | 4.07 | 3.30 |
| 8.0 | 65.40 | 36.87 | 23.48 | 16.38 | 11.97 | 9.23 | 7.23 | 5.93 |
| 10.0 | 102.03 | 57.42 | 36.67 | 25.55 | 18.70 | 14.45 | 11.30 | 9.25 |
| 12.5 | 159.70 | 89.85 | 57.40 | 39.98 | 29.27 | 22.62 | 17.68 | 14.50 |

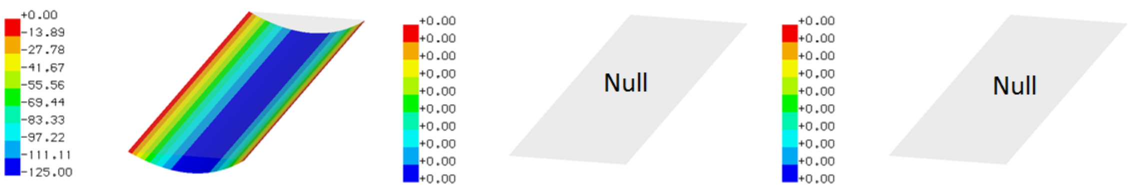
*Stiffeners should be used for spans in excess of 1100 mm to avoid excessive deflections.*

**Figure 5**: Extract from the SDM for the plate being considered

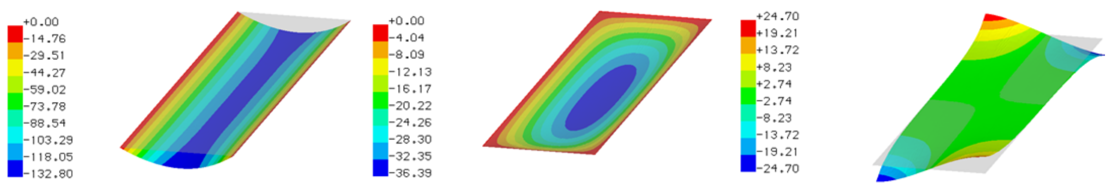
The 'ultimate load capacity' for the plate is 36.67kPa, as highlighted in Figure 5. The table does not refer to the width of the plate so it can reasonably be assumed that the mathematical model being adopted in the SDM is that of beam theory. However, the value shown is identical to the value already calculated for the pressure to cause first yield and so it is clear that the phrase 'ultimate load capacity' is spurious and misleading; if the practising engineer took the SDM value at face value then he/she would be underestimating the strength of the plate be some 50%!

## Mathematical Model - Plate

The theoretical solution for this particular plate configuration was worked out by Levy over a century ago and is published in, amongst others, Timoshenko's text on 'The Theory of Plates and Shells', [2], c.f., Figure 4(a). The moment fields for the beam and plate representations are compared in Figure 6.




(a) Beam representation




(b) Plate representation (Levy)

**Figure 6:** Cartesian moment fields (Nm/m for 1kPa pressure) - $m_x$, $m_y$ *and* $m_{xy}$

It is seen from Figure 6 that whereas the moment field for the beam is one-dimensional, that for the plate is fully two-dimensional. The beam representation has a maximum moment of equal value across the centre line whereas the plate representation shows maximum moments at the centre of the free (unsupported) edges and the pressure for first yield has already been calculated as 36.7kPa.

The two-dimensional moment field for the plate representation (which is driven by Poisson's ratio – anticlastic bending) leads to a pressure of 34.5kPa for a Poisson's ratio of 0.3.

As such, when the member is correctly represented with the mathematical model for a plate, then the moment to cause first yield is some 6% lower than that achieved with a beam model. Whilst the plate model does provide, more or less, the same solution as the beam model when the aspect ratio is small it soon begins to differ with increasing aspect ratio converging to a non-conservative 6% for aspect ratio greater than about 2 as shown in Figure 7.

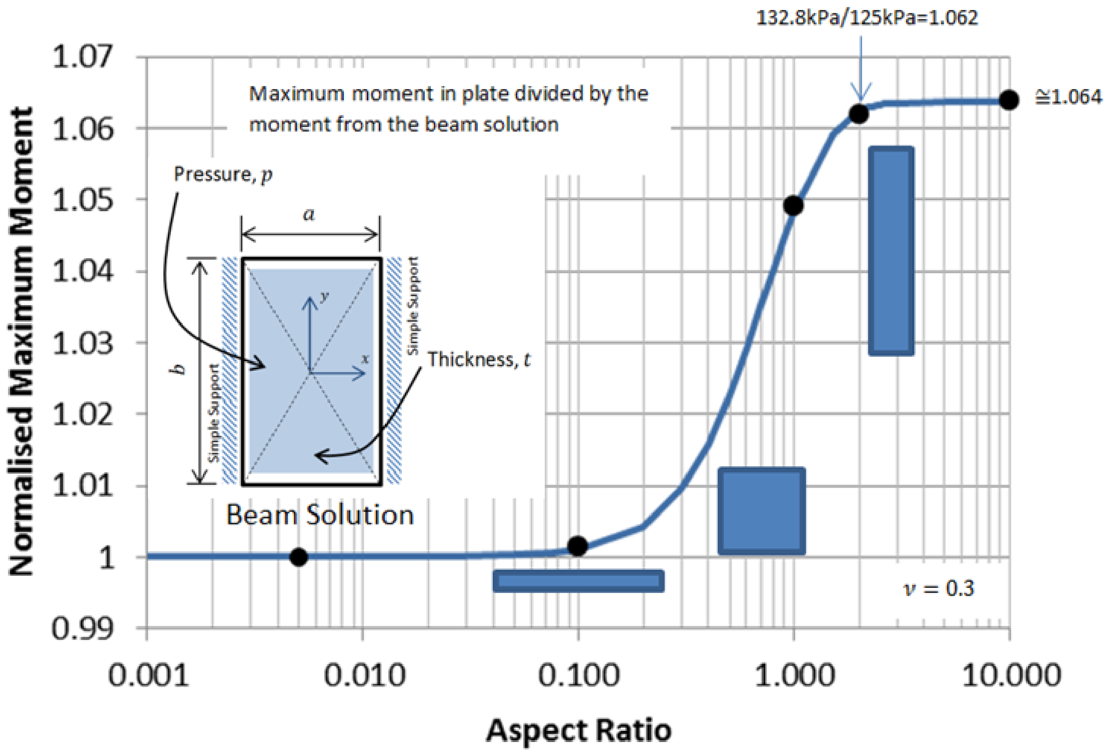



**Figure 7**: Maximum elastic moment for Levy

Thus, it has been shown that the beam representation is 6% non-conservative. Is that of concern? Does one consider the SDM (and others) to be doing the practising engineer a disservice given that the Levy solution has been known for at least 100 years?

It will have been spotted that the beta values at the centre of the plate for an aspect ratio of 0.5 were highlighted in the reproduction of the Timoshenko table in Figure 4. In undertaking this work RMA discovered that the value for *My* was incorrect, [3]. It is likely that this is a typographical error since the correct value is 0.0120 rather than 0.0102. The error means that the transverse moment at the centre of the plate is non-conservatively specified as some 15% below the correct value. This would not be of major concern for a steel plate since the maximum moment is elsewhere. However, if the engineer used this value for sizing the transverse reinforcement in a RC slab, then this might cause a problem.

## Finite Element Model of the Plate

The Levy solution, which is theoretically exact, provides the practising engineer with an ideal opportunity to verify has/her finite element software. The finite element solution, whilst approximate for coarse meshes, should converge, with mesh refinement, to the theoretical solution. This is demonstrated to be the case in Figure 8 where the results for regular meshes of lower and higher order CFE plate elements are used to predict the moment to cause first yield.

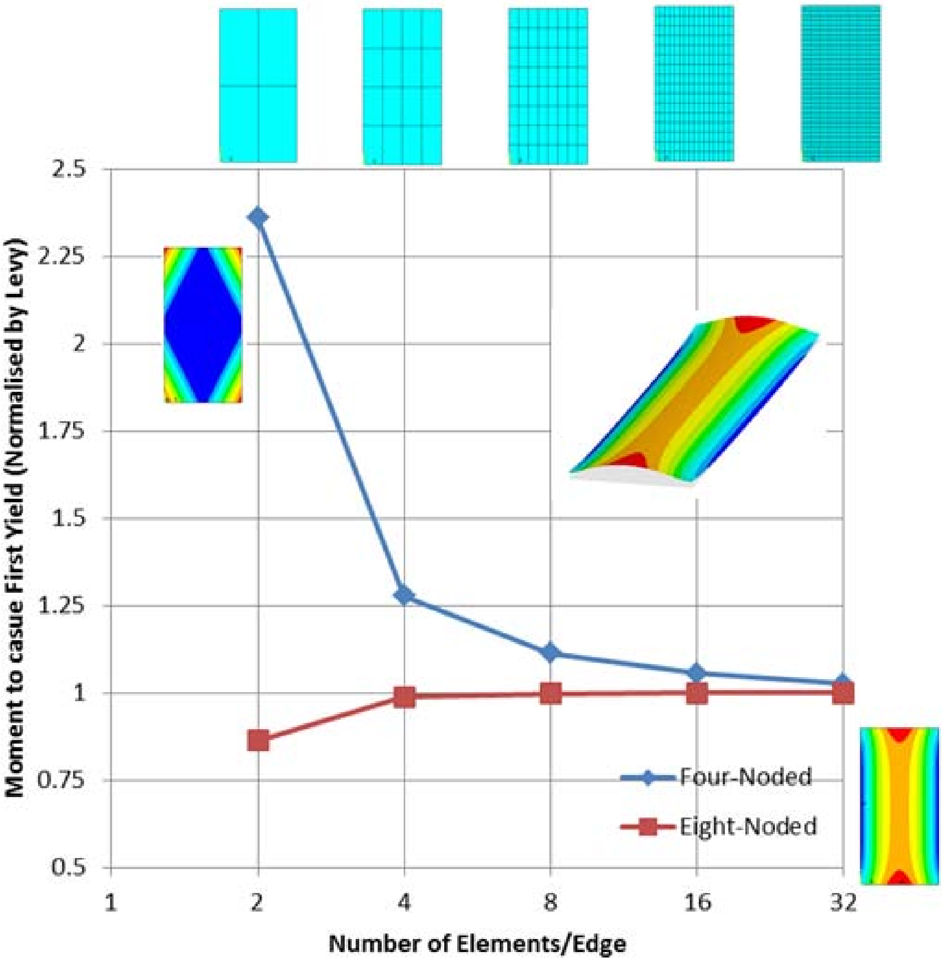



**Figure 8**: Convergence of CFE results to the Levy solution

As expected both lower and higher order elements converge to the theoretically exact Levy solution, the higher order element converging significantly more rapidly than the lower order element. Having conducted such a software verification study, the engineer is well placed to assume that the software is sound. Further, if he/she considers another problem of a similar nature, for example with a patch load over just a portion of the plate, which no longer possesses a known theoretical solution, then faith that finite element system will converge to the (unknown) theoretical solution has been developed.

It should be noted, with respect to Figure 8, that depending on the order of the element, the moment converges from either below (higher order) or above (lower order) the theoretical solution. This is an important point to note since the direction of convergence for point quantities in a finite element model (such as the maximum moment of Figure 8) cannot be guaranteed. This is contrary to the advice set out in the fib Model Code for Concrete Structures 2010, which states that:

**'The internal stresses are lower, compared with an exact solution'**

http://www.ramsay-maunder.co.uk/downloads/The%20fib%20Model%20Code%20for%20Concrete%20Structures%202010.pdf

The MC2010 code is simplistic in this respect and simply incorrect.

## Yield Criteria for RC Slabs and Steel Plates

The yield criterion appropriate for RC slabs is different to that which is appropriate for steel plates. The appropriate criterion for RC slabs is the Nielsen or square yield criterion whereas for steel plates it is the von Mises or elliptical criterion.

The appropriateness of a particular yield criteria for a particular material comes from the proposal of a mathematical model and then validation of this model with results observed from physical examples, c.f., Figure 1. Whilst not a full and detailed validation of the square criterion for RC slabs, the example shown in Figure 9 goes a long way to demonstrating the appropriateness of the square criterion.

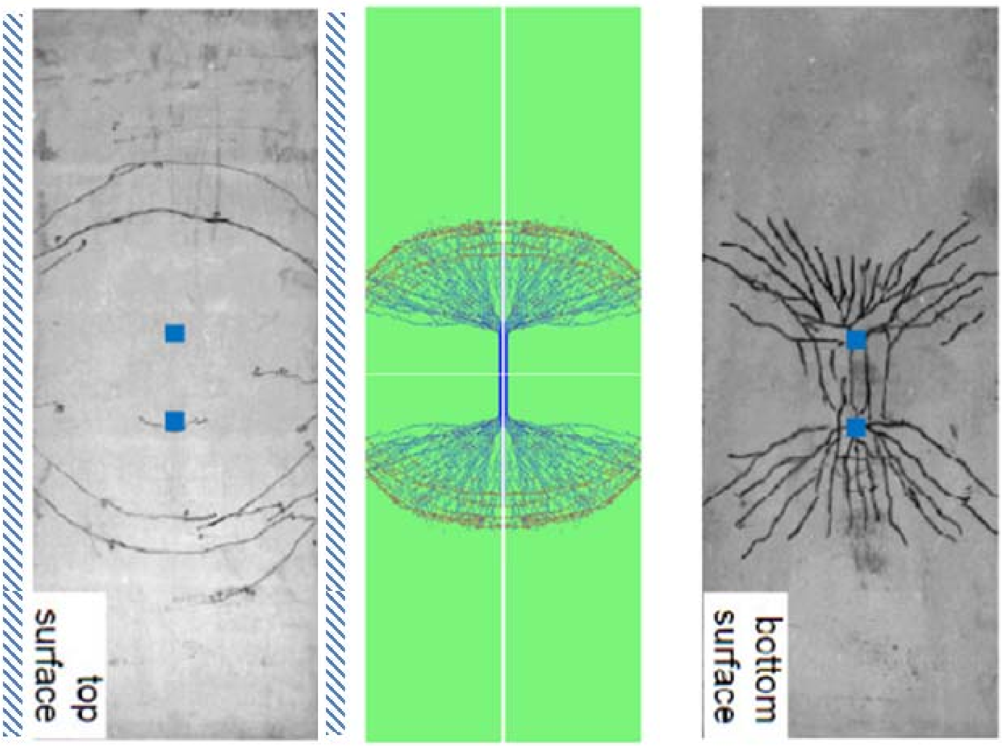



**Figure 9**: Quantitative verification of the square criterion for RC slabs

In a similar manner, validation of the elliptical criterion for steel plates was provided in 1932 by Taylor & Quinney who wanted to establish which of the Tresca or von Mises criteria were more appropriate for various ductile metals. The results, shown in Figure 2, demonstrated, unambiguously, that the von Mises criterion that is most appropriate.

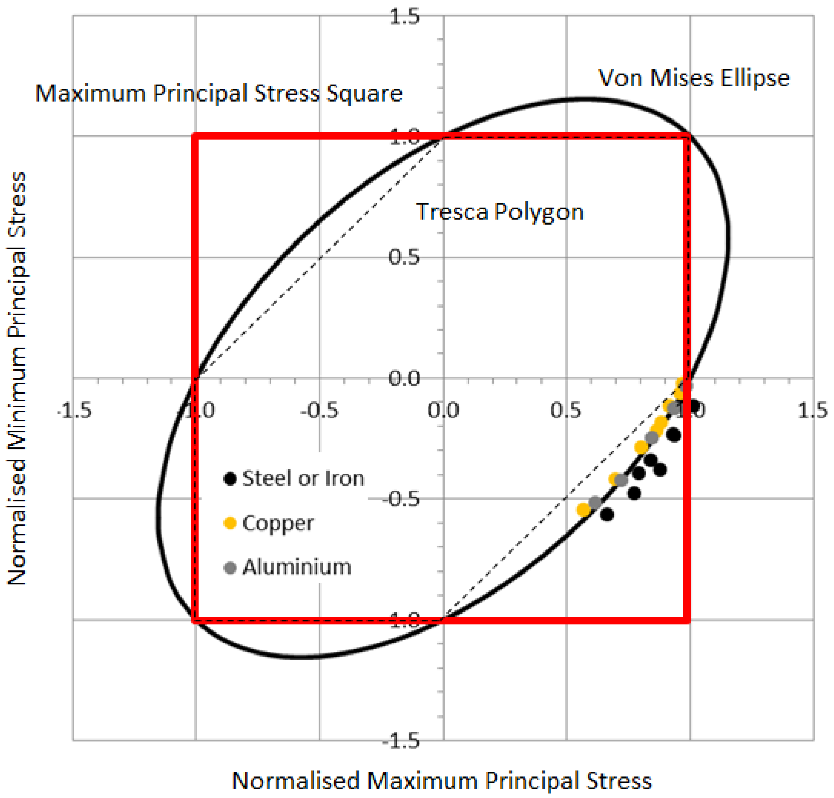



**Figure 10**: Quantitative verification of the elliptical criterion for steel plates

## Limit Analysis

Limit analysis is a technique for arriving directly at the collapse solution for a structure without having to increment along the load path, as would be required using conventional finite element techniques. It assumes a rigid-perfectly plastic material model and requires that the material be sufficiently ductile. The theoretically exact solution, like that for a linear elastic strength of materials problem, satisfies all the static and kinematic conditions. There are also, as described above for finite element analysis of linear elastic structures, two formulations which approximate the theoretical solution in different manners, either satisfying the kinematic conditions at the expense of the static conditions or *vice versa* as shown in Table *.

**Table 3**: Limit analysis formulations

| | Attributes | Kinematics | Statics | Techniques | Application |
|---|---|---|---|---|---|
| **Upper Bound** | Unsafe | Strong | Weak | Yield Line | Assessment |
| **Exact** | | Strong | Strong | | |
| **Lower Bound** | Safe | Weak | Strong | EFE | Design |

## Yield Line Technique for RC Slabs (Upper Bound)

The yield line technique began life as a hand calculation method for predicting the collapse load of RC slabs. Implicitly it makes use of the square yield criterion. This section will present how, over the years, this method has been automated so that analysis may now be conducted using computers and the landing slab example of Figure 11 will be used as a vehicle for this expedition.

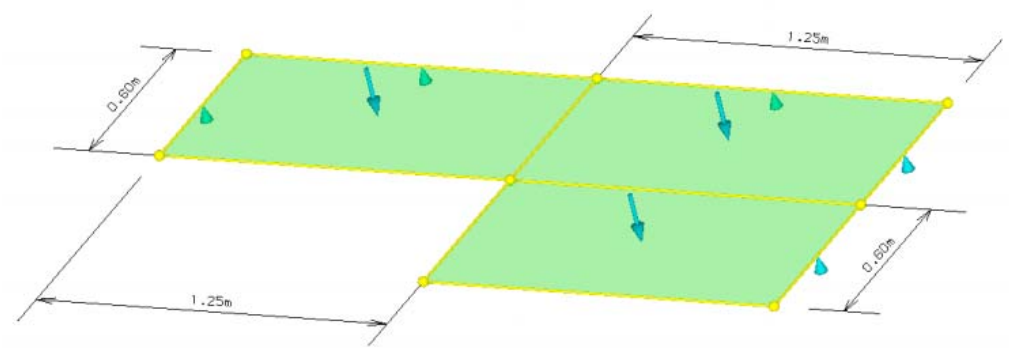
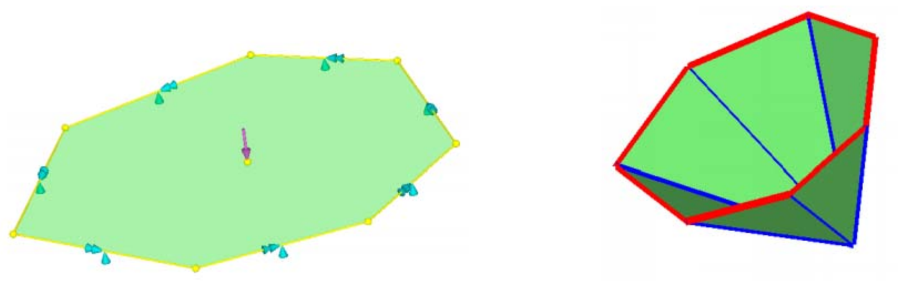



**Figure 11**: Landing Slab

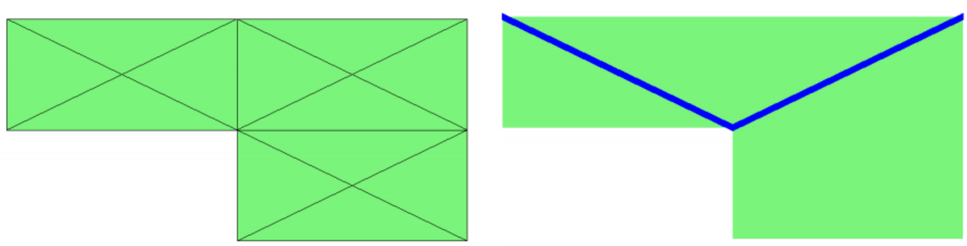
In computerising the yield line method, a finite element mesh of rigid triangular elements is adopted and each of the element edges is assumed a potential yield line. For example, the simple mesh shown in Figure 12 of seven elements has a simple yield line mechanism, which looks like an (inverted) pitched roof.



(a) Model (b) Yield lines and displaced shape

**Figure 12**: Basic fan mechanism

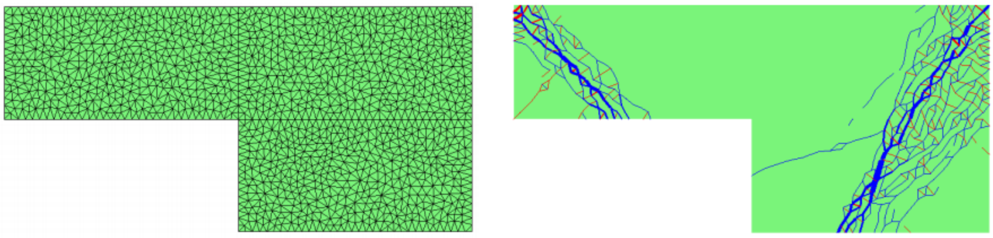
A coarse mesh of the landing slab, Figure 13, comprises four such mechanisms and, using an optimisation procedure, where the collapse load is minimised, the collapse mechanism of Figure 13 is obtained.



(a) Mesh (b) Yield Line Pattern

**Figure 13**: Results from 1997 ($\lambda$=5.86)

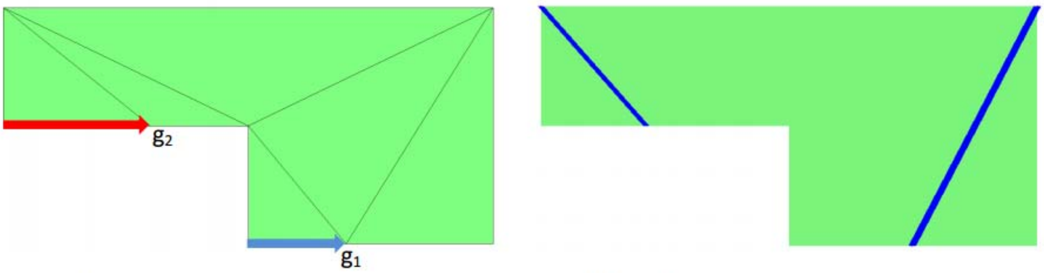
A more refined and unstructured mesh allows more possible collapse mechanisms and the critical collapse mechanism, although 'fuzzy' can begin to be discerned – Figure 14.



(a) Mesh (b) Yield Line Pattern

**Figure 14**: Results from 2011 ($\lambda$=5.47)

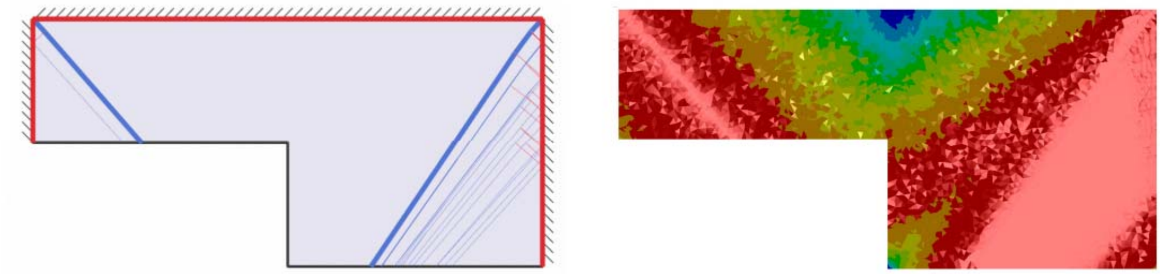
Having obtain an idea of the critical collapse mechanism then a coarse mesh which includes this mechanism can be constructed and geometric optimisation performed to further lower the collapse load, i.e., hoe in on the theoretical solution – Figure 14.




(a) Mesh (b) Yield Line Pattern

**Figure 14**: Results from 2011 – Geometrically Optimised ($\lambda$=4.38)

The yield line technique has been successfully further automated using the DLO method. This approach can rapidly produce high quality yield line solutions that are reliably close to the theoretical solution as illustrated in Figure 15.



(a) LimitState:SLAB ($\lambda$=4.21) (b) RMA:EFE ($\lambda$=4.20)

**Figure 15**: State of the Art Results from 2014

Using an EFE approach, as discussed in the next section, a dual solution lower bound solution to the upper bound DLO solution, which enables a very close bounding of the theoretical solution thereby providing engineers with confidence that the software has recovered a good representation of the true solution.

## Equilibrium Finite Elements for RC Slabs (Lower Bound)

Unlike the yield line technique, an equilibrium approach such as adopted by EFE provides a lower bound estimate of the collapse load irrespective of the mesh adopted. It does this be ensuring that the moment fields are in strong equilibrium with the applied loads and that the moment field is always within the relevant yield criterion. Whilst Figure 12 provided a simple way in which to understand the computerisation of the yield line technique, a simple RC beam problem will be used to illustrate the lower bound approach.

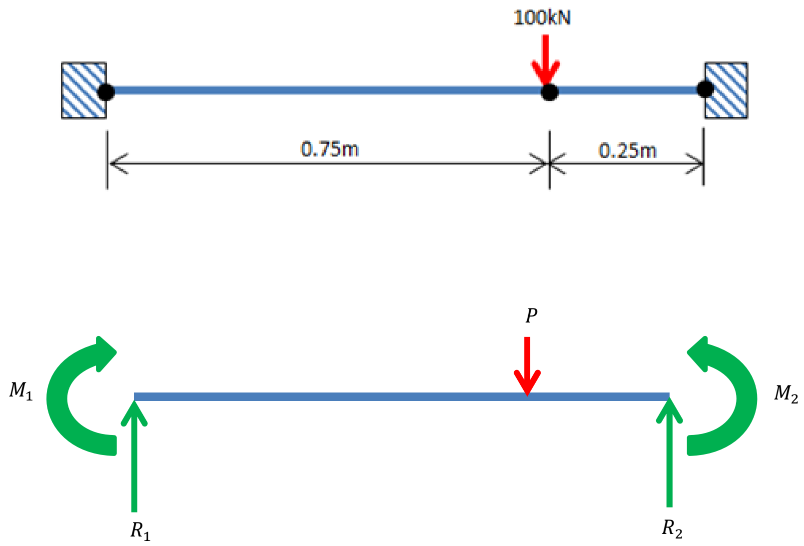



**Figure 16**: Fixed beam under point load and free-body diagram

The beam is twice statically indeterminate, i.e., there are two unknown moment reactions. The total moment field for this beam may then be considered as the sum of a particular solution, which satisfies equilibrium with the applied load, and two self-balancing or hyperstatic moment fields, which are in equilibrium with zero applied load as shown in Figure 17.

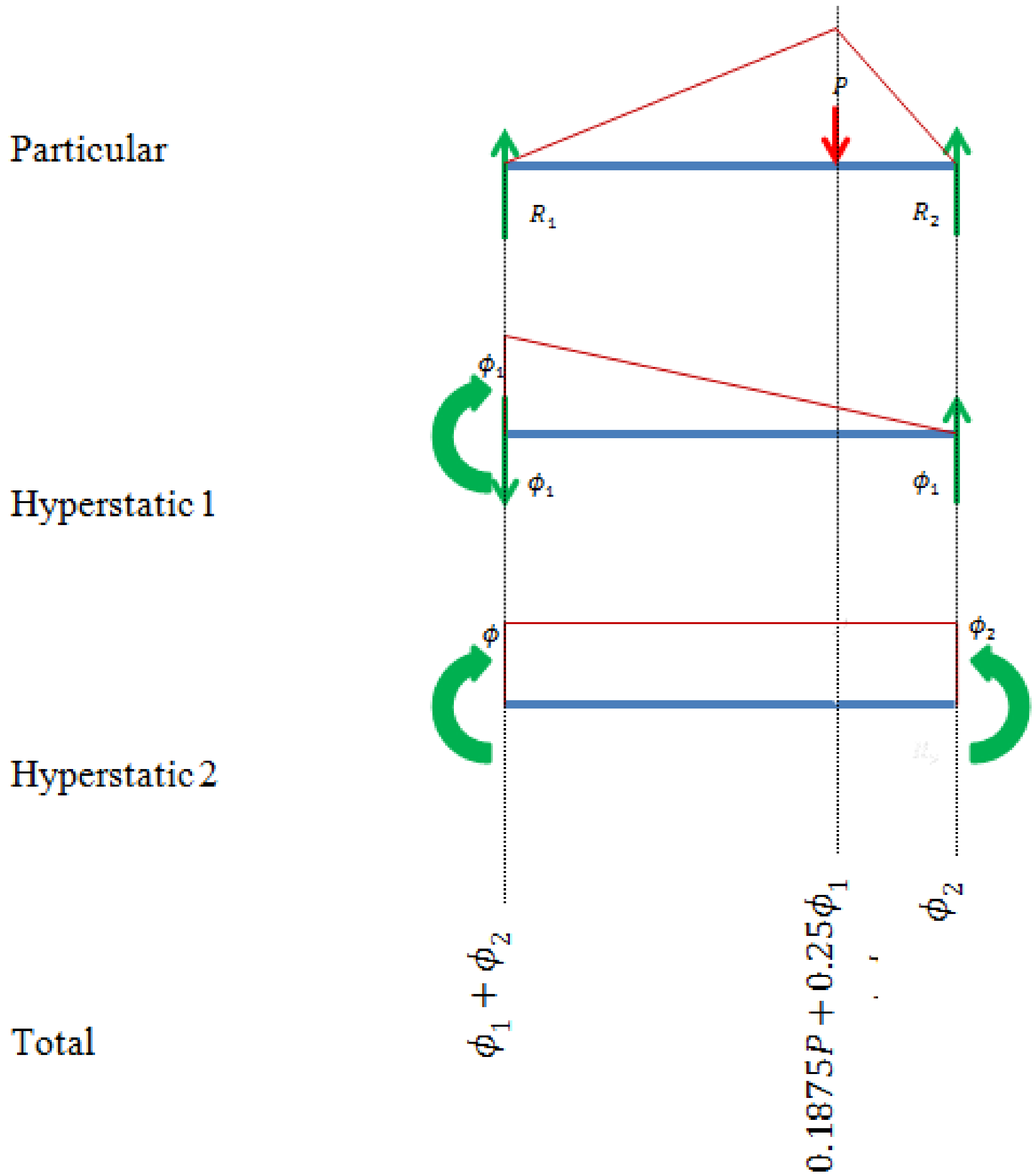



**Figure 17**: Particular and self-balancing hyperstatic moment fields

The moments at the two ends of the beam and under the point load can be collected together in matrix form and equated to, what we know, to be the collapse moments for this problem:

$$\begin{bmatrix} 0 & 1 & 1 \\ 0.1875 & 0.25 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} P \\ \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -m_c \\ +m_c \\ -m_c \end{Bmatrix} \tag{3}$$

If we choose a beam of $t$=0.01m thickness and yield stress of $S_y$=275MPa then the moment capacity is:

$$m_c = \frac{S_y \cdot t^2}{4} = 6875Nm/m \tag{4}$$

Three equilibrium solutions are shown for this problem in Figure 18. The elastic solution with the load to cause first yield of 32.6kN, the theoretical collapse solution with collapse load of 73.3kN and the particular solution scaled to develop a plastic hinge under the load (36.7kN).

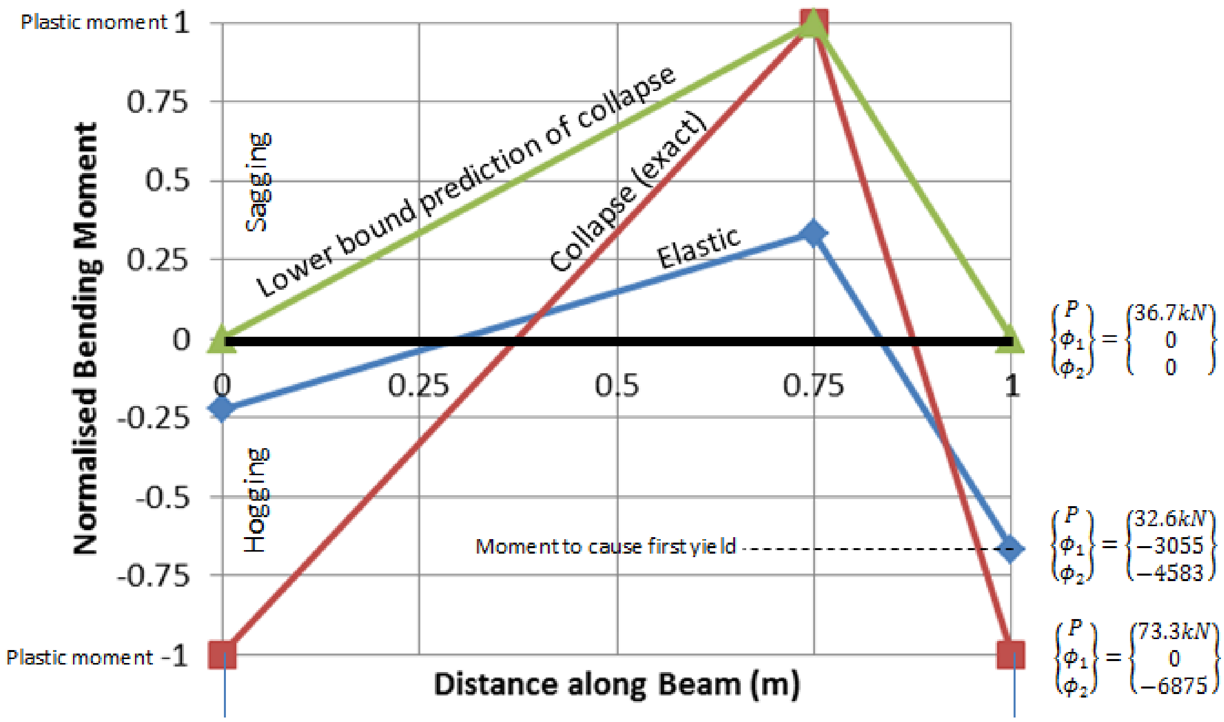



**Figure 18**: Equilibrium solutions, elastic, lower bound and theoretical collapse

As already discussed in this paper, for the safe design of a structure the engineer requires the stress or moment fields to balance the applied load so that he/she can then size reinforcement to withstand these moments, i.e., make the design sufficiently strong. The equilibrating moment fields provided by EFE then are perfect for the design of RC slabs. Consider the rectangular RC slab of Figure 19 which is uniformly loaded and simply supported on two adjacent sides.

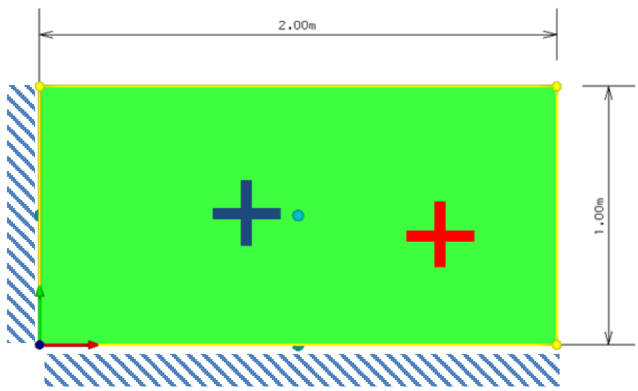
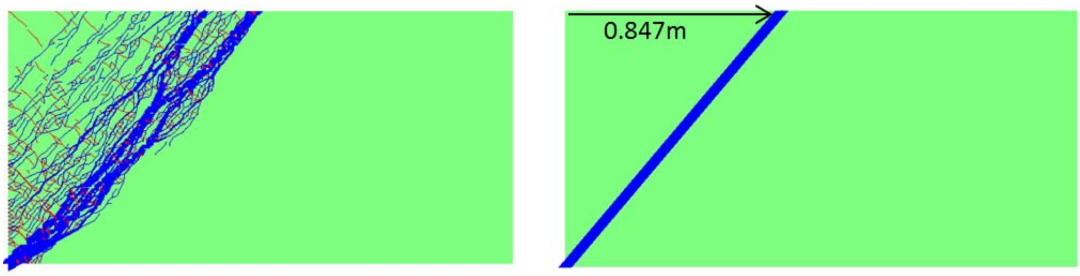



**Figure 19**: Uniformly loaded rectangular slab simply supported on two adjacent sides

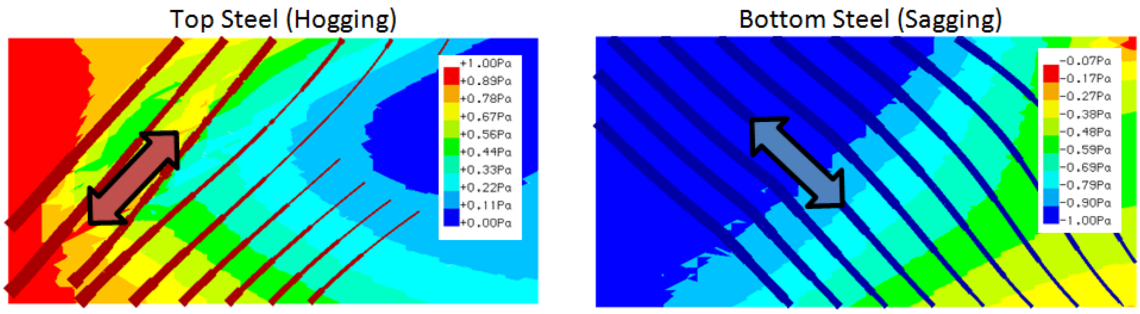
A yield line analysis was first conducted for this slab, in the manner outlines previously, using first a refined unstructured mesh to provide understanding of the critical collapse mechanism and then with a coarse mesh incorporating the critical mechanism upon which geometric optimisation could be performed. The results are shown in Figure 20.




(a) Refined unstructured mesh (30kN/m$^2$)    (b) Coarse optimised mesh (23.61kN/m$^2$)

**Figure 20**: Yield line results for RC slab

The next step was to analyse the slab in EFE and the lower bound results are shown in Figure 21. Note the very close agreement between the collapse loads from the upper and lower bound approaches.




Contours of hogging and sagging reinforcement utilisation are shown ranging between 0 and 1 for hogging and between 0 and -1 for sagging. Superimposed on these are the relevant hogging or sagging trajectories. These diagrams show the engineer where reinforcement steel is required, and the optimal direction for the reinforcement – parallel to the principal moment trajectories.

**Figure 21**: Initial design - results from EFE (23.6kN/m$^2$)

In Figure 21, the principal moment trajectories are shown. These show directly the optimal direction for the reinforcement and indirectly the required moment capacity of the reinforcement. It is clear from the trajectories that the optimal reinforcement should be oriented at about 45 degrees to the sides of the slab. If this is done then it is possible to remove two of the four layers of reinforcement completely. To confirm this intuitive leap, the reinforcement in EFE was rotated, and two layers removed and the result is shown in Figure 22.

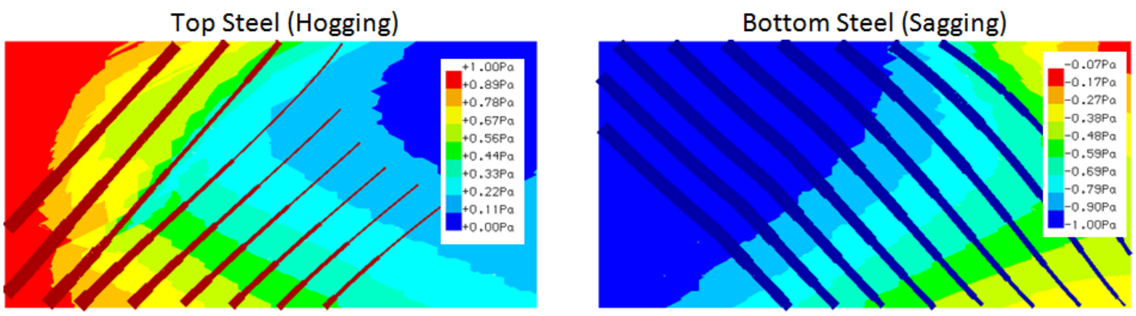



**Figure 22**: Optimised design – results from EFE (23.3kN/m$^2$)

A small but insignificant reduction in the collapse load is noted for this optimised design indicating that in this example rotating the steel placement can lead to significant reductions in the required amount of reinforcement, in this case a 50% reduction. It should be noted that the design above only considers the ULS condition. The SLS conditions of maximum deflection and cracking will lead to their own reinforcement requirements, which may mean adding back some of the steel here removed.

## Equilibrium Finite Elements for Steel Plates (Lower Bound)

The plate problem of Figure 4 is now reconsidered in terms of its plastic collapse load using the elliptical yield criterion. The collapse load varies with aspect ratio in the manner shown in Figure 23.

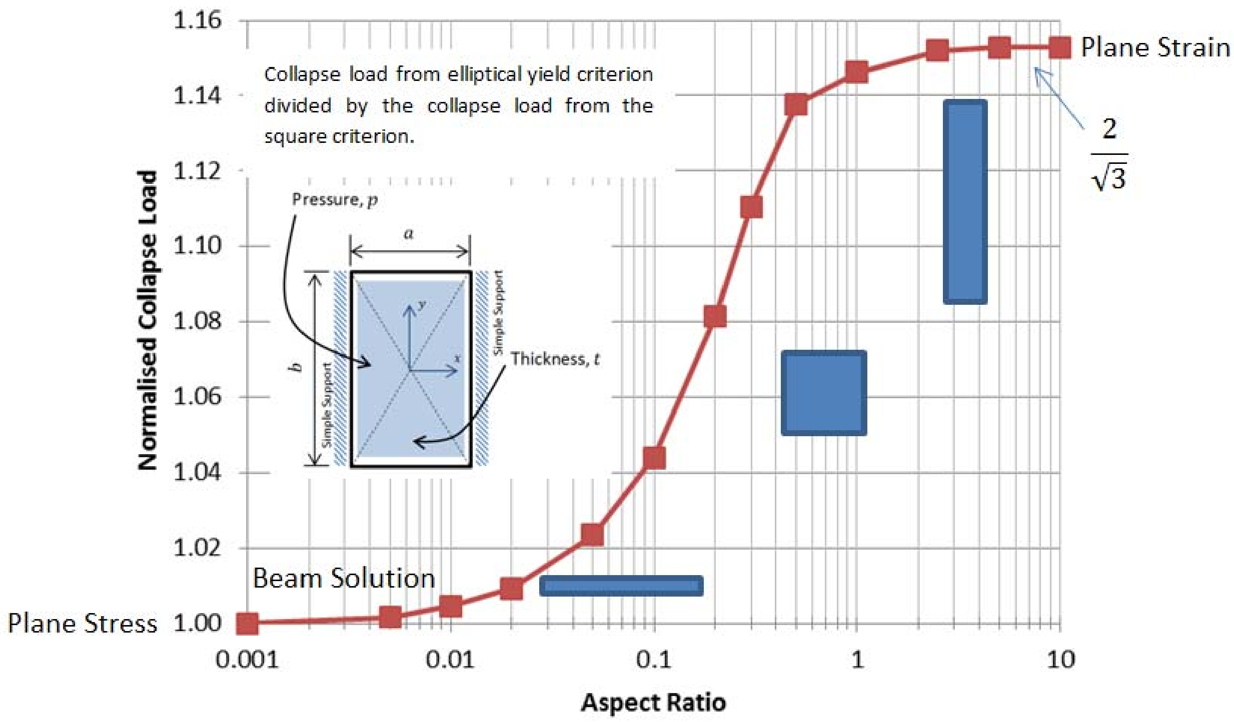



**Figure 23**: Collapse pressure from EFE as a function of aspect ratio

The collapse load from EFE is significantly greater than that predicted by beam theory and for the plate configuration considered, aspect ratio of 2, the utilisation contours are plotted in Figure 24. When EFE is used with the square yield criterion then the collapse load is 1.5 times the load to cause first yield for the beam.

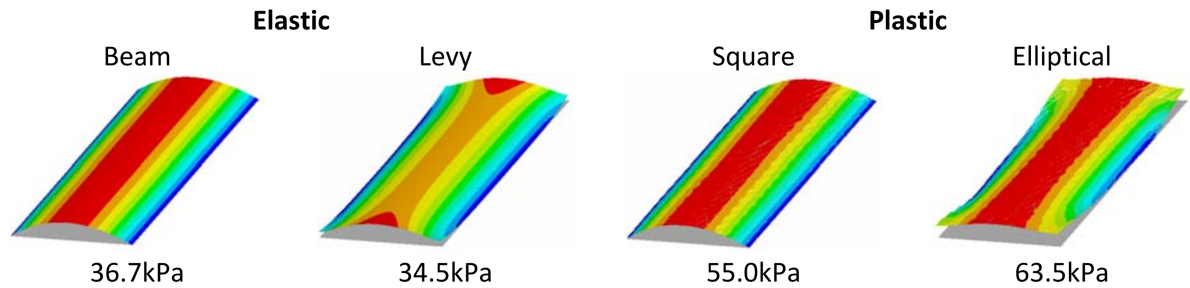
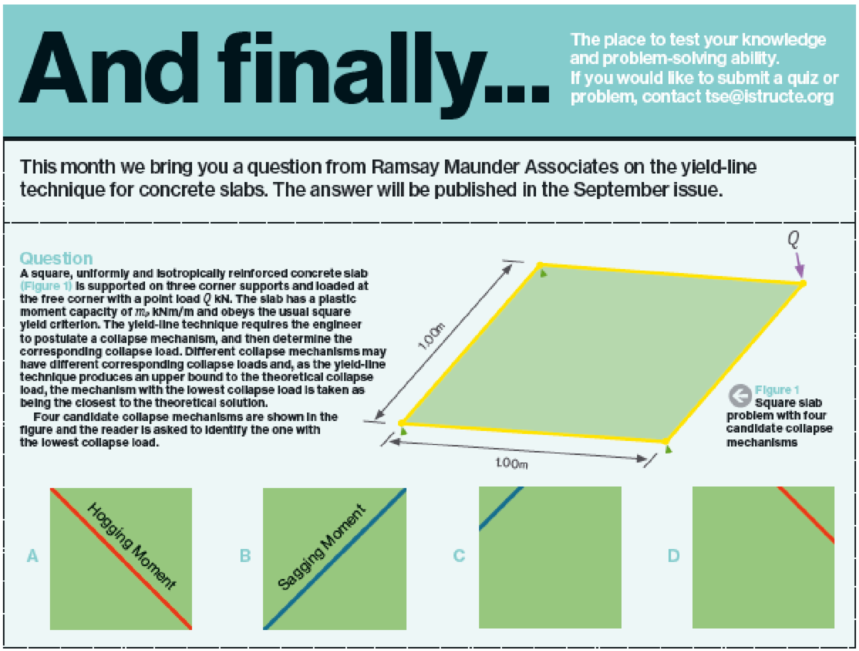



**Figure 24**: Contours of utilisation for the plate problem and collapse loads

The reason for the $2/\sqrt{3}$ factor between the collapse load for the square and elliptical criteria will be explained after first considering another example where using the elliptical criterion leads to a significantly lower collapse load than would be obtained with the square criterion. This problem is the same constant moment problem presented, by RMA, in the IStructE's 'And Finally …' section of August edition of The Structural Engineer magazine – see Figure 25. The reason for presenting this problem was that it exhibited very clearly a situation where multiple yield line patterns all with the same collapse load could be easily established. As presented, the problem was considered as an RC slab where the square yield criterion is appropriate. The collapse load was determined as twice the moment capacity of the slab, i.e., $Q = 2m_p$. If the elliptical criterion were used then the collapse load would have been factored down by $1/\sqrt{3}$.



# And finally...

The place to test your knowledge and problem-solving ability.
If you would like to submit a quiz or problem, contact tse@istructe.org

This month we bring you a question from Ramsay Maunder Associates on the yield-line technique for concrete slabs. The answer will be published in the September issue.

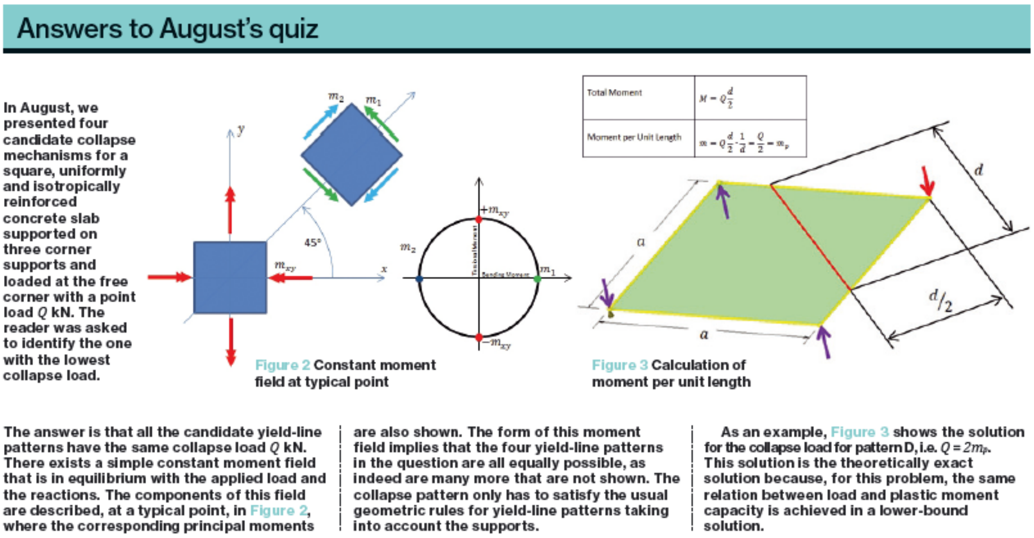
**Question**

A square, uniformly and isotropically reinforced concrete slab (Figure 1) is supported on three corner supports and loaded at the free corner with a point load $Q$ kN. The slab has a plastic moment capacity of $m_p$ kNm/m and obeys the usual square yield criterion. The yield-line technique requires the engineer to postulate a collapse mechanism, and then determine the corresponding collapse load. Different collapse mechanisms may have different corresponding collapse loads and, as the yield-line technique produces an upper bound to the theoretical collapse load, the mechanism with the lowest collapse load is taken as being the closest to the theoretical solution.

Four candidate collapse mechanisms are shown in the figure and the reader is asked to identify the one with the lowest collapse load.



Figure 1 Square slab problem with four candidate collapse mechanisms



## Answers to August's quiz

In August, we presented four candidate collapse mechanisms for a square, uniformly and isotropically reinforced concrete slab supported on three corner supports and loaded at the free corner with a point load $Q$ kN. The reader was asked to identify the one with the lowest collapse load.



Figure 2 Constant moment field at typical point

Figure 3 Calculation of moment per unit length

The answer is that all the candidate yield-line patterns have the same collapse load $Q$ kN. There exists a simple constant moment field that is in equilibrium with the applied load and the reactions. The components of this field are described, at a typical point, in Figure 2, where the corresponding principal moments are also shown. The form of this moment field implies that the four yield-line patterns in the question are all equally possible, as indeed are many more that are not shown. The collapse pattern only has to satisfy the usual geometric rules for yield-line patterns taking into account the supports.

As an example, Figure 3 shows the solution for the collapse load for pattern D, i.e. $Q = 2m_p$. This solution is the theoretically exact solution because, for this problem, the same relation between load and plastic moment capacity is achieved in a lower-bound solution.

**Figure 25**: RMA's 'And Finally …' question and answer [4, 5]

The two examples of the plate from Figure 4 and the 'And Finally …' plate show the two extremes of difference in collapse load between the square and elliptical yield criteria and this is depicted graphically in Figure 26.

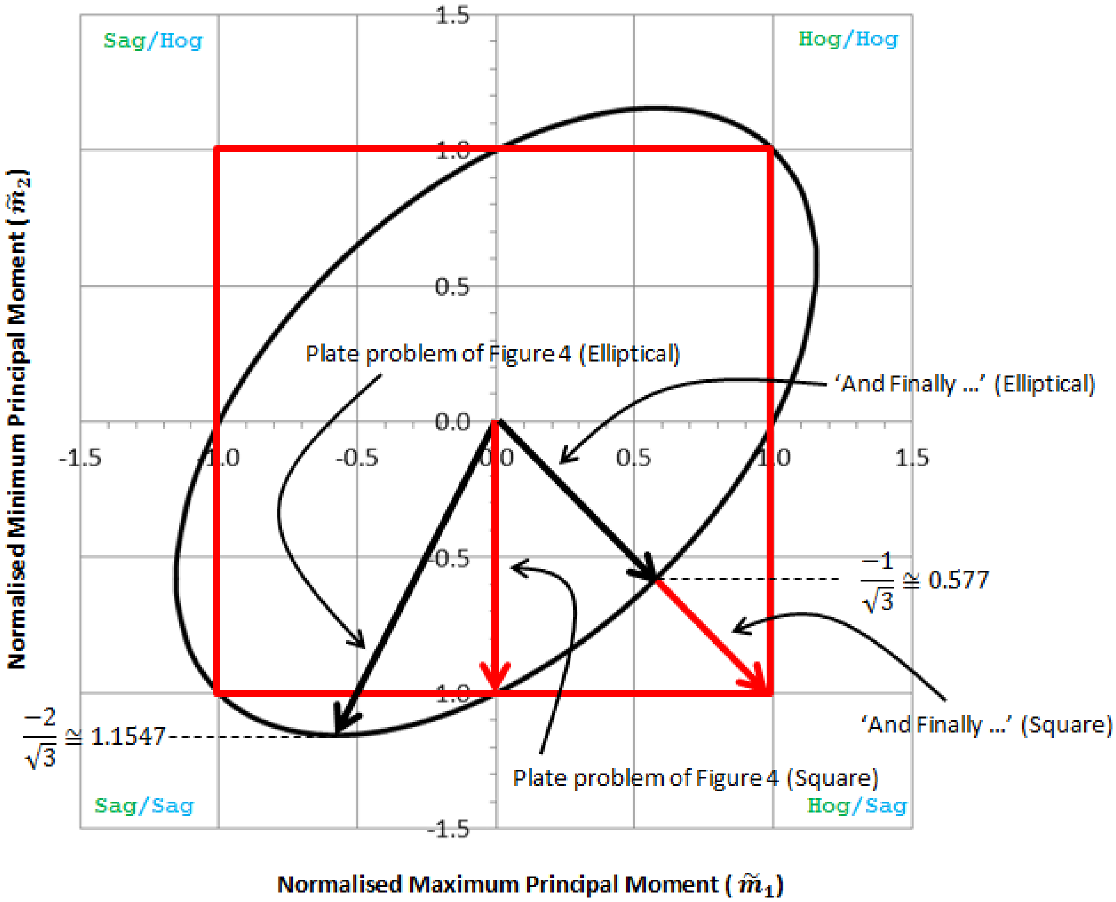



**Figure 26**: Collapse of two plate problems in principal moment space

In terms of the load factor $\lambda$ with subscripts $e$ and $s$ for the elliptical and square yield criteria respectively, then:

$$\frac{1}{\sqrt{3}}\lambda_s \leq \lambda_e \leq \frac{2}{\sqrt{3}}\lambda_s \tag{5}$$

## Closure

In this presentation, we have aimed to illustrate some of the research and development work undertaken at Ramsay Maunder Associates over recent years. With Equilibrium Finite Elements (EFE), it has been possible to glean considerable insight into the way RC slabs and steel plate members behave as they collapse. It is hoped that some illumination has been shed on the following points:

- Simulation governance and its components Verification & Validation
- Characteristics of different FE formulations for linear elasticity – CFE & EFE
- Essential requirement of Equilibrium for structural design

- Equilibrium is only achieved with a CFE model as the mesh is refined
- Risk of using an inappropriate mathematical model (beam versus plate representation)
- Verification of FE systems by modelling a problem with a known theoretical solution
- Validation of different yield criteria for RC concrete and steel members
- Characteristics of different formulations for limit analysis
- Yield line techniques, how they have been computerised
- The danger of unsafe collapse load prediction through yield line analysis
- Lower bound, safe, limit analysis through an EFE formulation
- EFE in the design of RC slabs – rotate reinforcement and get rid of 50% of it!
- EFE in the assessment of steel plates – significant differences in collapse loads through different yield criteria.

## References


[1] Davison. B, and Owens, G.: Steel Designers' Manual, The Steel Construction Institute, 6th Edition.

[2] Timoshenko S.P. and Woinowsky-Krieger S.: Theory of Plates and Shells (2nd ed.), New York, USA, McGraw-Hill, 1989.

[3] Ramsay, A.C.A., and Maunder, E.A.W.: An error in Timoshenko's "Theory of Plates and Shells", The Structural Engineer, IStructE, June 2016.

[4] Ramsay Maunder Associates, 'And Finally …', The Structural Engineer, IStructE, August 2016.

[5] Ramsay Maunder Associates, 'And Finally …', The Structural Engineer, IStructE, September 2016.


Note: The Ramsay Maunder Associates website, www.ramsay-maunder.co.uk, it a good resource for further information on the topics of this presentation.