Engineering Tripos Part IIA

Module 3D4

Structural Analysis and Stability

Handout 2

Prof Chris Burgoyne (cjb19@)

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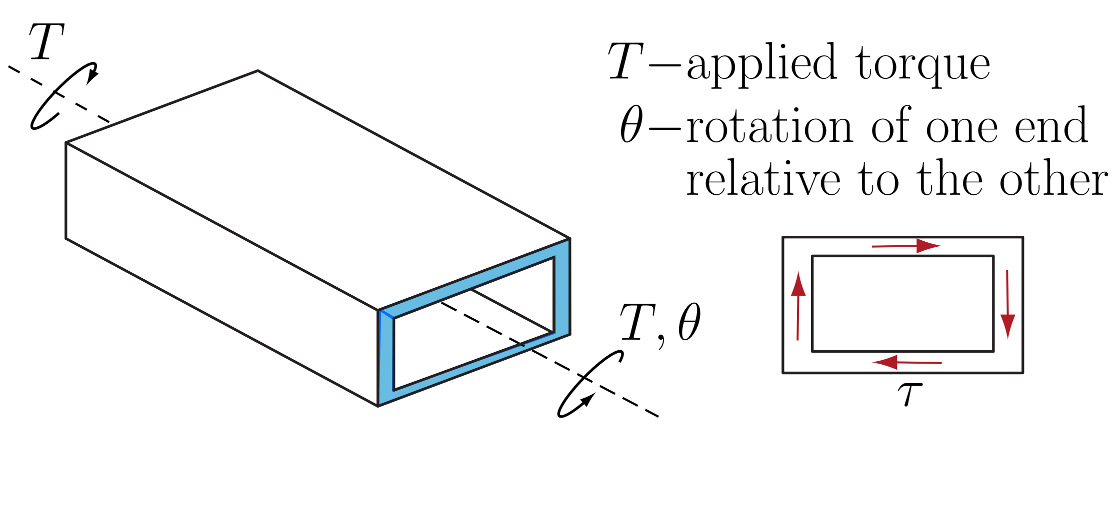
3 Torsion of beams

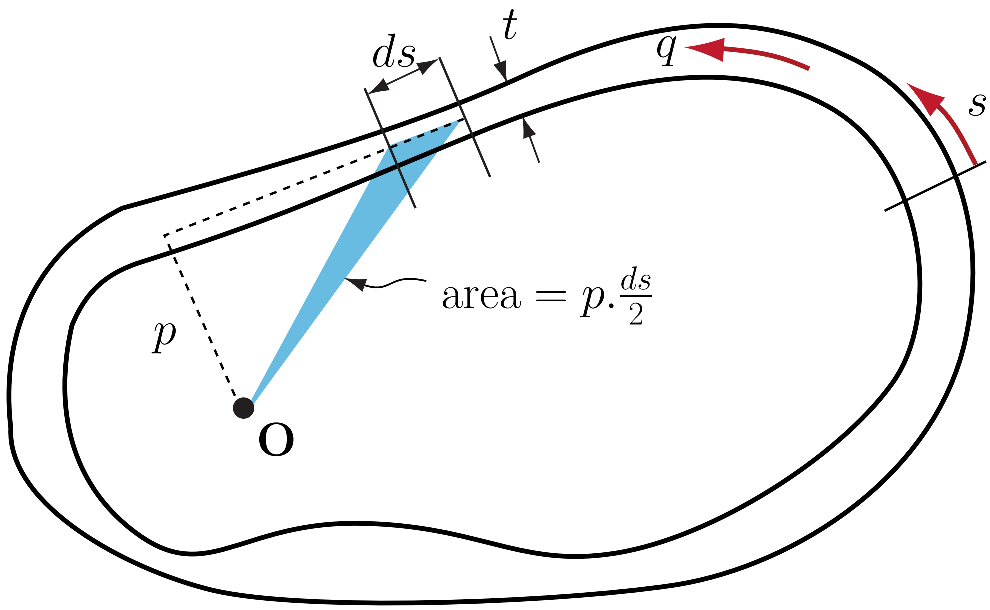
Torsion in beams arises from shear loads whose points of application do not coincide with the shear centre of the section.

In general, the solution of torsion problems is complex particularly in case of solid sections of arbitrary shape. However, the analysis of circular shafts and closed or open thin-walled tubes is relatively straightforward.

3.1 Revision: Torsion of thin-walled tubes

We consider a closed-wall section with (possibly variable) thickness t and applied torque T.





It can be shown that the shear flow $q = \tau \cdot t$ has to be constant around the tube in order to satisfy equilibrium.

Now we consider the torque caused by the infinitesimal areal element with length ds about an arbitrary point O

Torque due to $ds \cdot q$: $dT = ds \cdot q \cdot p$

Total torque: $T = \int dT = q \int pds = 2qA$

Moreover from Part IB, we know the stiffness relationship between torque and rate of twist

$$T = G\frac{4A^2}{\oint \frac{ds}{t}}\theta' \quad \text{with} \quad \theta' = \frac{d\theta}{dz}$$

where θ is the rotation of one end of the beam relative to the other (i.e. twist). The above equation leads to the definition of a section-specific St Venant's torsion constant J

$$J = \frac{4A^2}{\oint \frac{ds}{t}}$$

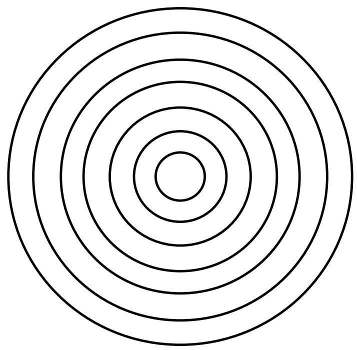
so that

$$T = GJ\theta'$$

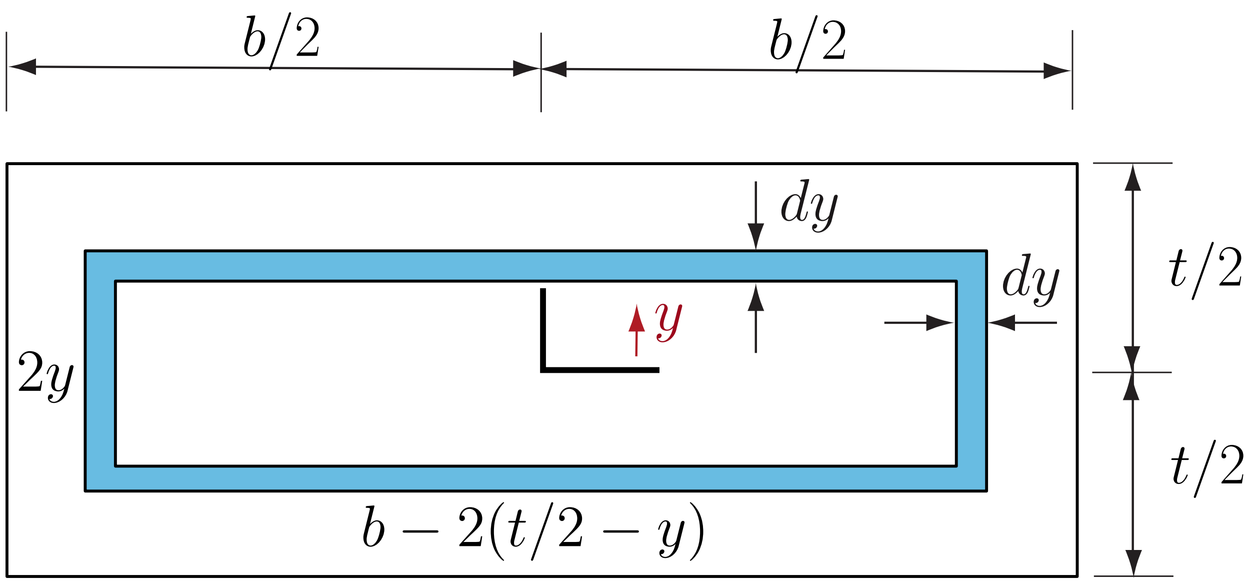
where GJ is known as the torsional rigidity of the section.

3.2 Torsion of solid sections

In the following an approximate method for analysing solid sections is introduced. First, recall that under torque cross-sections rotate only as a rigid body (as seen from the centroidal axis). Therefore, we can 'nest' tubes inside one another and add their effects.



Consider the rectangle section



Take a thin tube with the thickness dy, then sum the effects of all thin tubes to get their total effect.

Enclosed area by the thin tube $A = 2y(b - t + 2y)$

$$\oint \frac{ds}{dy} = \frac{2(b-t+2y)}{dy} + \frac{4y}{dy}$$

Now integrate over all the strips in order to compute the torsion constant J

$$\begin{aligned} J &= \int_0^{t/2} \frac{4A^2}{\oint \frac{ds}{dy}} \\ &= \int_0^{t/2} \frac{4 \cdot 4y^2(b-t+2y)^2}{2(b-t+2y)+2y} \cdot dy \\ &= \left(\frac{bt^3}{3} - \frac{t^4}{12}\right) = \frac{bt^3}{3} \ (\text{if } b \gg t) \end{aligned}$$

Finally, the relationship between the torque and rate of twist for the rectangle section reads

$$T = GJ\theta' = G\frac{bt^3}{3}\theta'$$

This analysis is not accurate since tubes ***DO*** *interact but error is in the* t^4 *term so the overall result is correct. A more general derivation makes use of the* *Prandtl stress function* *(see, e.g., 3C7)).*

The maximum <u>shear stress</u> τ_{max} occurs on the outer surface of the rectangle.

To compute τ_{max} consider the outer layer dy which carries the torque dT

$$dT = G\theta' \frac{4A^2}{\oint \frac{ds}{dy}} = 2qA \qquad \text{(see Pages 53 and 54)}$$

$$\Rightarrow q = \tau_{max} dy = G\theta' \cdot \frac{2A}{\oint \frac{ds}{dy}}$$

$$= G\theta' \frac{2bt}{2(b+t)} dy$$

$$\Rightarrow \tau_{max} \approx G\theta' \cdot t \ (\text{if } b \gg t\ )$$

This can be related to the torque, using the relationship between torque and rate of twist derived on Page 56.

$$T = G\theta' . \frac{bt^3}{3}$$

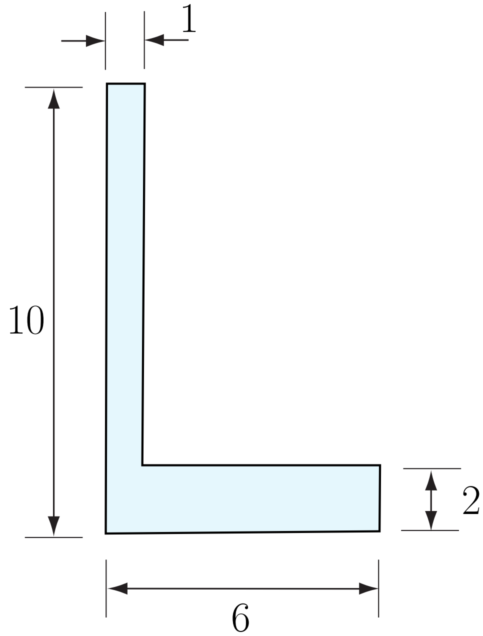
so that

$$\tau_{max} = \frac{3T}{bt^2}$$

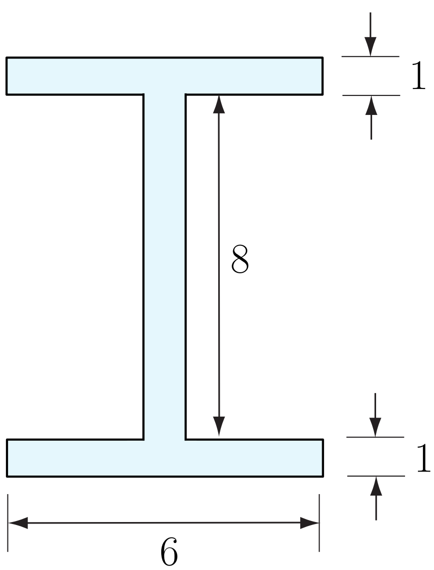
This value applies for $b \gg t$. See, for example, Timoshenko for more accurate values.

The torsion constant for built-up sections can be computed as the sum of torsion constants of the component sections (but see Junction effects on Page 64.

$$J = \sum \frac{1}{3} b t^3$$



$$\begin{aligned} J &= \frac{1}{3} \cdot 10 \cdot 1^3 + \frac{1}{3} \cdot 6 \cdot 2^3 \\ &= \frac{58}{3} = 19.3 \; (L^4) \end{aligned}$$



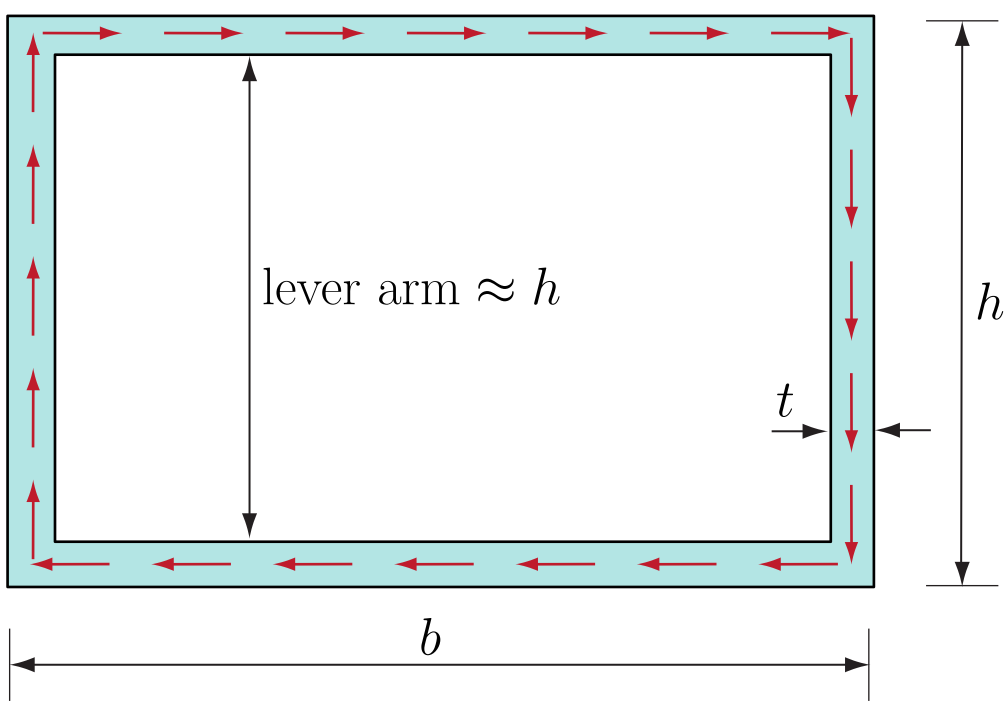
$$\begin{aligned} J &= \frac{1}{3} \cdot 6 \cdot 1^3 \cdot 2 + \frac{1}{3} \cdot 8 \cdot 1^3 \\ &= \frac{20}{3} = 6.7 \; (L^4) \end{aligned}$$

3.3 Torsion of closed versus open sections

Under torsional loads closed as well as open section beams twist and develop internal shear stresses. However, the way how each resists torsion is different.

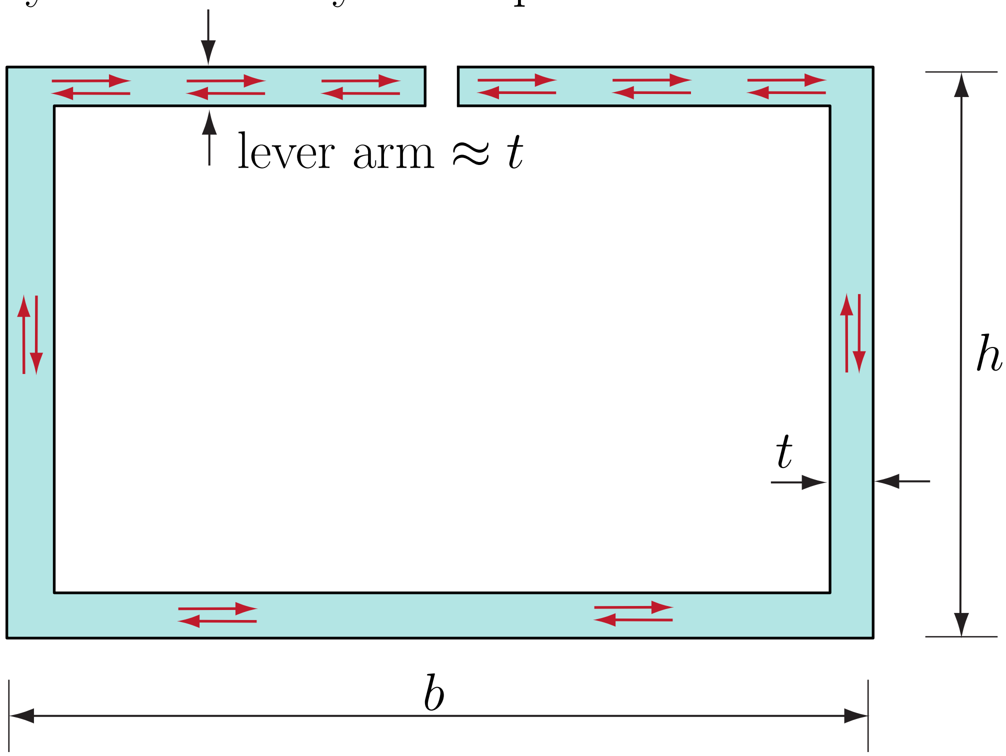
Note that a pure torque applied to a beam section has to produce a shear stress system with a zero resultant force (i.e. $\int \tau \, dA = 0$).

Closed sections In a closed section the shear stress system can develop in a continuous path around the cross section.



$$J \approx \frac{4(bh)^2}{2\frac{(b+h)}{t}} = 2\frac{(bh)^2 t}{b+h} = \mathcal{O}(b^3 t)$$

Open sections In an open section the shear stress system can only develop within the thickness of walls.



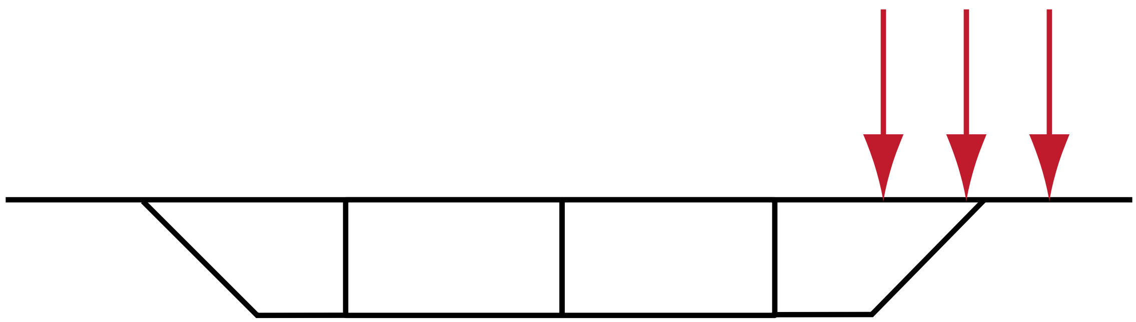
$$J \approx \sum \frac{1}{3} b t^3 = \frac{1}{3}(2ht^3 + 2bt^3) = \mathcal{O}(bt^3)$$

As a result closed sections are much stiffer than open sections in torsion.

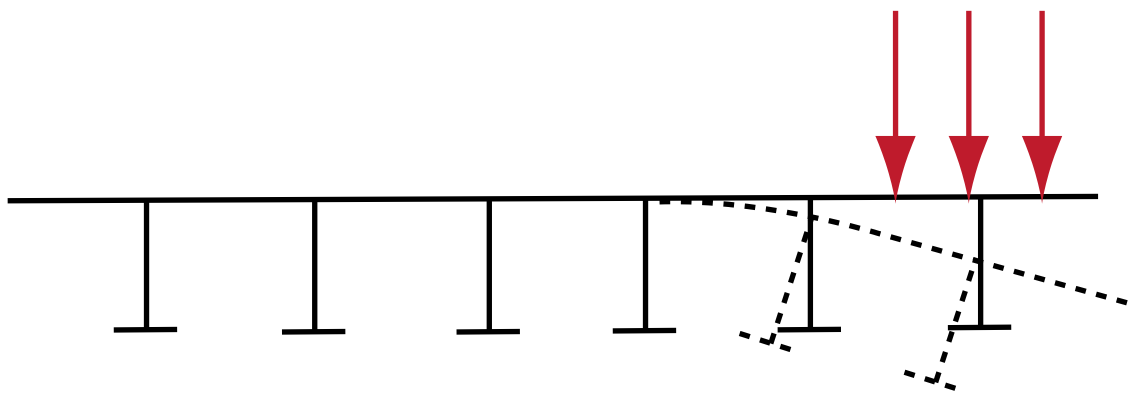
Practical considerations

Box beams are a widely used closed cross-section in bridge engineering. They are used with the purpose

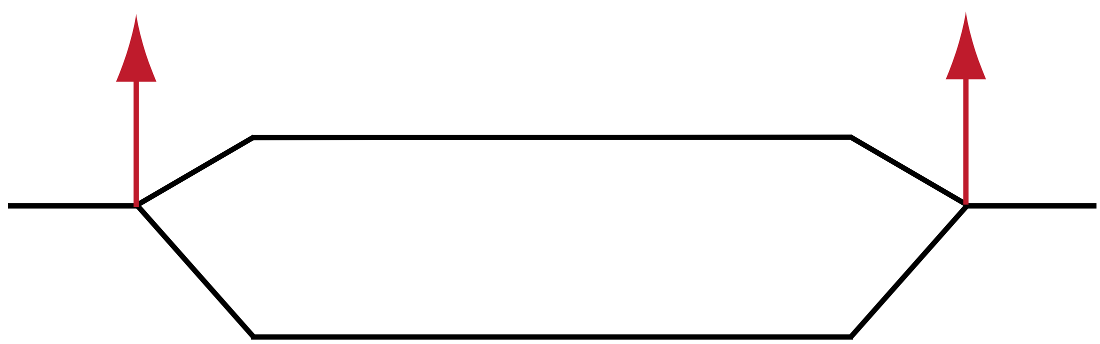
- to distribute loads



(They are better than plate girders for distributing loads)

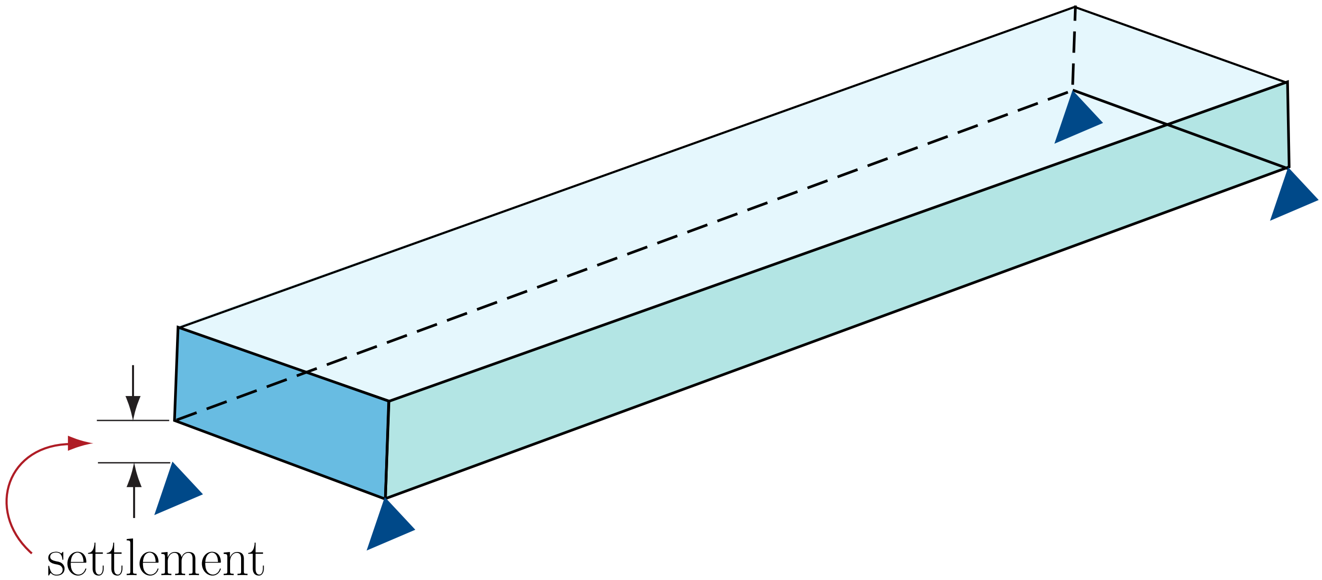


- to provide torsional stiffness to resist aerodynamic flutter



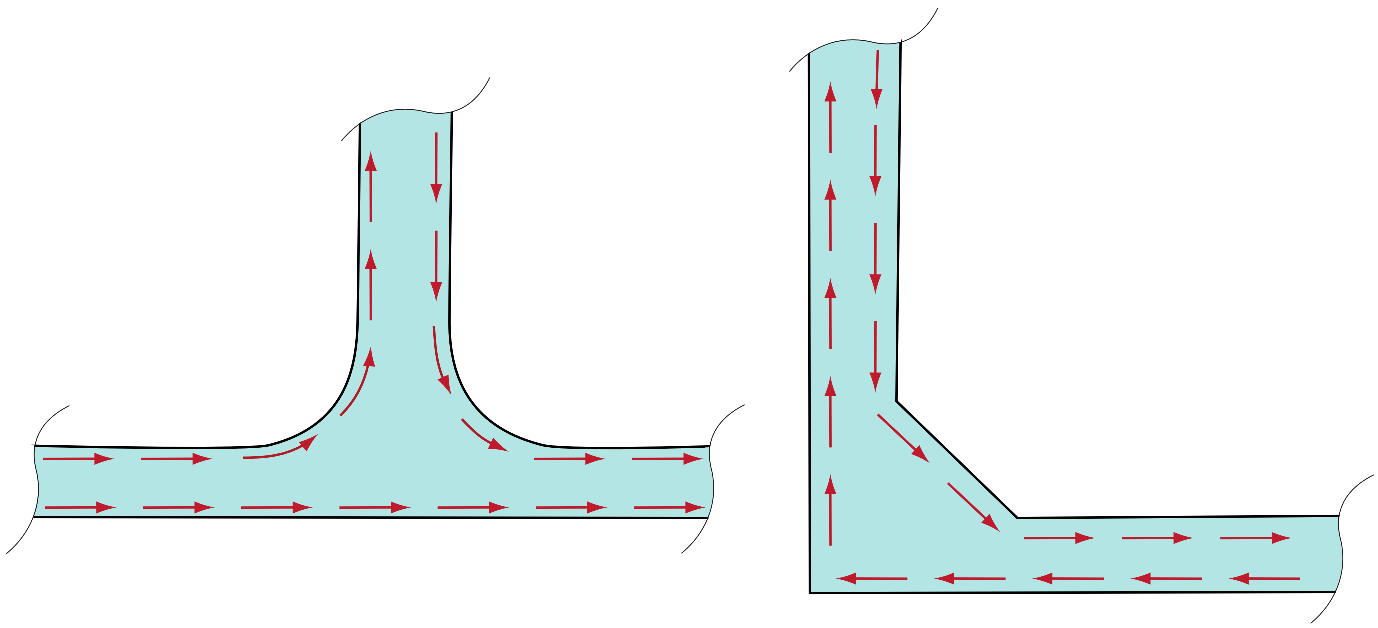
(e.g. Severn & Humber suspension bridges)

But their relative stiffness with respect to torsion can lead to difficulties with indeterminate supports.



Beam ends up rocking on <u>two</u> supports.

3.4 Effects of junctions on torsional stiffness

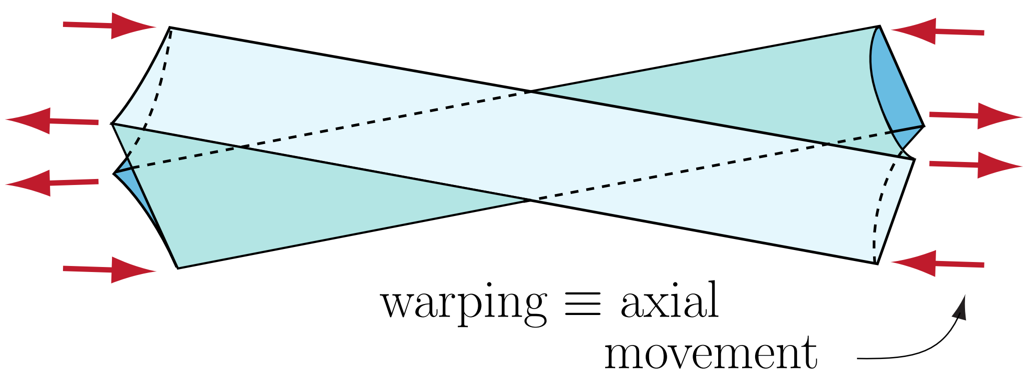


The section thickness is significantly increased in junction regions since the torsional stiffness varies as t^3. A major part of the torsional stiffness of rolled steel beams and cast concrete beams comes from this region.

3.5 Restrained warping torsion

Although the shapes of closed and open sections remain undistorted during torsion as seen from the centroidal axis, they do not remain plane.

Consider the deformation of a beam with rectangular section during torsion.



The section remains rectangular when twisted (viewed from the centroidal axis). However, it warps out of its plane. The analysis discussed in the previous section assumes section is <u>free to warp</u>, so no axial stresses are set up.

The formulation introduced in the previous section is called St Venant torsion. It is only correct if

- the ends of the beam are free to warp
- the torque is constant along the centroidal axis

But, if torque varies along the centroidal axis or if ends offer axial restraint, adjacent sections will *try* to warp by different amounts, so axial strains and stresses will be set up.

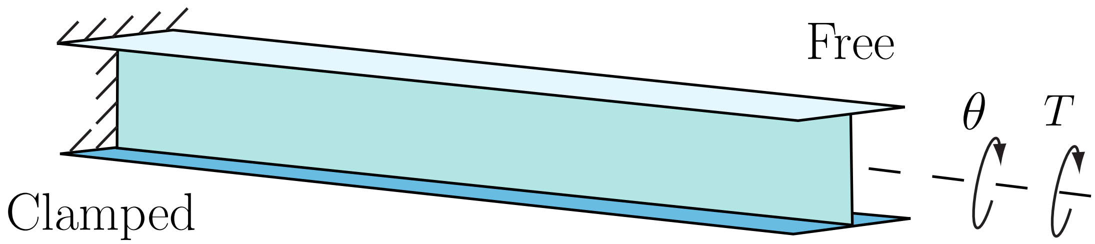
These will have associated shear stresses, which will carry some of the torque.

This is known as the restrained warping torsion.

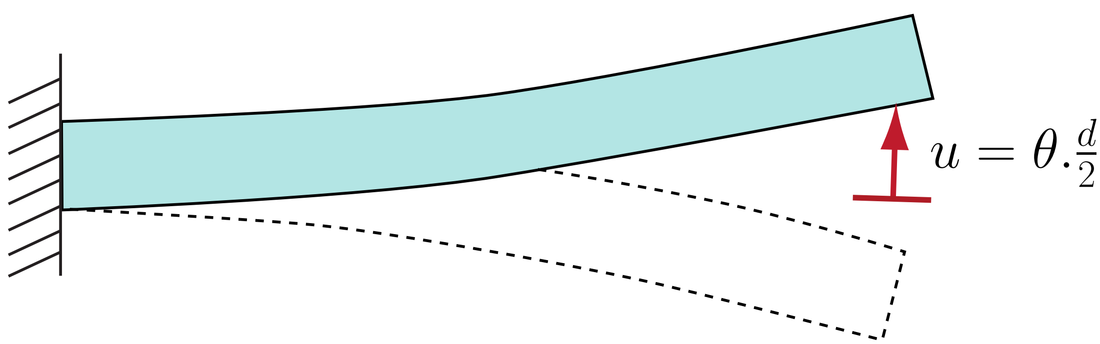
Remark: A few sections do not warp under torsion. These include solid and hollow circular sections and square box sections of constant thickness.

Illustrative example for restraint warping torsion

Consider an I-beam mounted as a cantilever, with torque applied at the free end.



Due to torque the two flanges will bend in opposite directions.



Note that d is the height of the section and θ is the twist that is yet to be determined.

Curvature of top flange in its own plane is

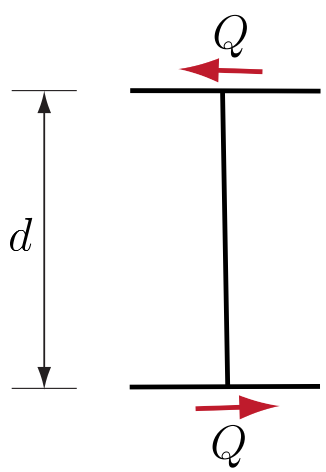
$$\kappa = -\frac{d^2u}{dz^2} = -\frac{d^2\theta}{dz^2}.\frac{d}{2} = -\theta''\frac{d}{2}$$

The associated bending moment in the flange is

$$M = EI_f\kappa = -EI_f\theta''\frac{d}{2}$$

where I_f is the second moment of area of the flange. The shear force in the flange follows from

$$Q = \frac{dM}{dz} = -EI_f\theta'''\frac{d}{2}$$



Similar force in other flange but of opposite sense.

Hence the torque carried is

$$Qd = -EI_f\frac{d}{2}\theta'''d = -E\Gamma\theta'''$$

$$\text{with} \qquad \Gamma = I_f\frac{d^2}{2} \qquad (\text{units } L^6)$$

The total torque carried by the St. Venant and the restrained warping component is

$$T = GJ\theta' - E\Gamma\theta'''$$

where Γ is known as the restrained warping torsion constant

The foregoing (ordinary) differential equation has to be solved for determining the twist θ. The related boundary conditions are

At root:

- No twist $\Longrightarrow \theta = 0$
- Flange built-in $\Rightarrow \dfrac{du}{dz} = 0 \Longrightarrow \theta' = 0$

At free end:

- No moment in flange $\Rightarrow EI_f \frac{d}{2}\theta'' = 0 \Longrightarrow \theta'' = 0$

Aside: The general equation for restrained warping torsion, not just for this example, is

$$T = GJ\theta' - E\Gamma\theta'''$$

with the two possible boundary conditions

$\theta' = 0$ for warping restrained

$\theta'' = 0$ for free to warp

For the cantilever I-beam considered here the solution of the differential equation is determined will be of the form

$$\theta = \frac{T}{GJ}(z + c) + ae^{-\alpha z} + be^{\alpha z}$$

Substituting this into the differential equation yields

$$\alpha^2 = \frac{GJ}{E\Gamma}$$

$$\implies \alpha = \frac{1}{\lambda} = \sqrt{\frac{GJ}{E\Gamma}}$$

λ has dimensions of length and is a property of the section

The constants in the ansatz are determined using the boundary conditions

$$\theta'' = 0 \text{ at } z = L \;\Rightarrow \frac{b}{a} = -e^{\frac{-2L}{\lambda}}$$

$$\theta' = 0 \text{ at } \; z = 0 \;\Rightarrow \frac{T}{GJ} - \frac{a}{\lambda}(1 - \frac{b}{a}) = 0$$

$$\Rightarrow a = \frac{T}{GJ}\frac{\lambda}{(1 + e^{-2\alpha L})}$$

$$\theta = 0 \text{ at } z = 0 \;\Rightarrow c = -\lambda\frac{(1 - e^{-2\alpha L})}{(1 + e^{-2\alpha L})}$$

Finally, the solution can be written as

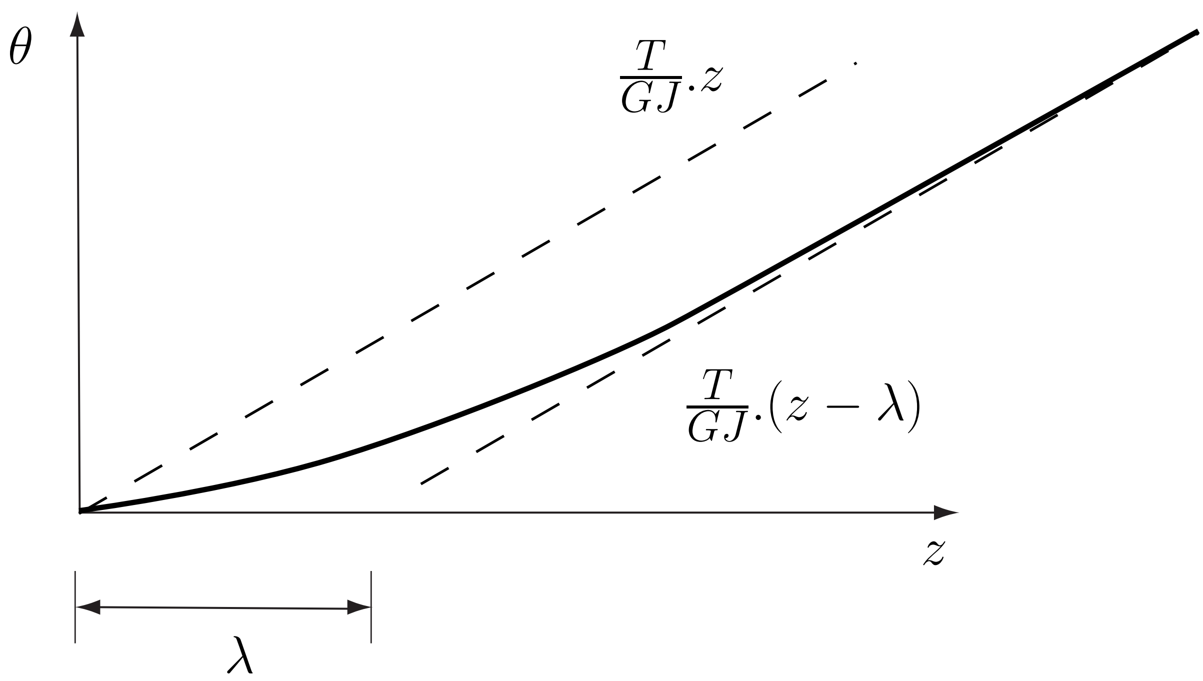
$$\theta = \frac{T}{GJ}(z + c) + \frac{T}{GJ}\lambda\frac{(e^{-\frac{z}{\lambda}} - e^{\frac{(z-2L)}{\lambda}})}{(1 + e^{-2\alpha L})}$$

In the long beam limit (i.e. $L \gg \lambda$) the constants approach

$$c \to -\lambda \qquad b \to 0 \qquad a \to \frac{T\lambda}{GJ}$$

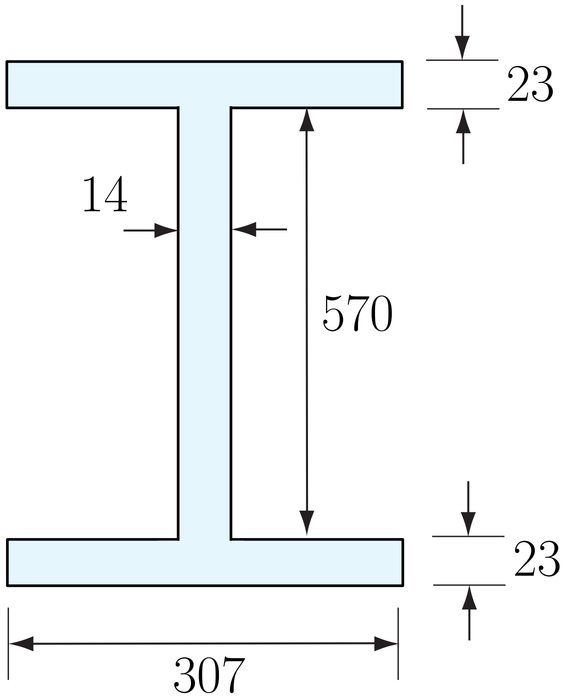
so that the solution approaches to

$$\theta = \frac{T}{GJ}(z - \lambda(1 - e^{-\frac{z}{\lambda}}))$$



Clamping one end of the beam reduces its effective length for torsion by $\lambda = \sqrt{\frac{E\Gamma}{GJ}}$ (even for constant torque).

Importance of warping restraint for practical sections Example $610 \times 305 \times 179kg$ U.B. (see data book)



$$J = \frac{1}{3}\sum bt^3 = 3{\cdot}10^6\,mm^4$$

$$I_f = \frac{307^3 \cdot 23}{12} = 55{\cdot}10^6\,mm^4$$

$$\therefore \Gamma = \frac{I_f d^2}{2} = 55 \cdot 10^6 \frac{(570+23)^2}{2} = 9.75 \cdot 10^{12}\,mm^6$$

$$\therefore \lambda = \sqrt{\frac{E\Gamma}{GJ}} = \sqrt{\frac{210}{81} . \frac{9.75 \cdot 10^{12}}{3 \cdot 10^6}} = 2900\,mm$$

Typical span/depth ratio $\approx$ 20 so span likely to be $\approx$ $12m$.

$\Rightarrow \lambda \approx \frac{L}{4}$ warping restraint will be significant .

3.6 Summary

- Warping restraint normally significant for open sections.
- Angle sections warp slightly but shear centre at junction of the legs so shear forces have no lever arm, so warping can be ignored.
- Closed sections warp, but warping displacements smaller and GJ higher $\Rightarrow \lambda$ much lower .
- Circular sections do not warp at all because of symmetries .
- Calculation of shear centre and $E\Gamma$ is not trivial. Most data books do not give them.
- Best summary of values 'Buckling Strength of Metal Structures' Bleich. (FD10) (attached below).
  - Theory given in 'Theory of Elastic Stability' Timoshenko & Gere Sect. 5.3 (FD14).

Data for position of Shear Centre and value of Restrained Warping Torsion constant.

Taken from Buckling Strength of Metal Structures by F. Bleich. McGraw-Hill, 1952 (CUED Library FD10)

38. Properties of Sections

The value of Γ and of the coordinates x_0 and y_0 of the center of shear were determined for a number of sections and are listed here. The results were simplified considerably by expressing Γ, x_0, and y_0 in terms of the area and moments of inertia of the entire section wherever possible.

The values for the angle- and T-sections were derived by the Goodier theory, using Eq. (216) for the warping. The location of the shear centers for these sections was assumed to be at the intersection of the center lines of the legs. This is only approximately true, but the distance of the center of shear from this point is always a very small fraction of the thickness of the leg and can be neglected. The value Γ for these sections is small, and for many applications $\Gamma = 0$ may be used.

Angle, Equal Legs (Fig. 47):

$$x_0 = 0, \qquad y_0 = e, \qquad \Gamma = \frac{A^3}{144} \tag{225}$$

where A is the cross-sectional area of the angle.

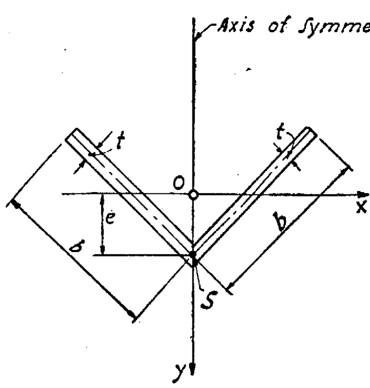


Fig. 47

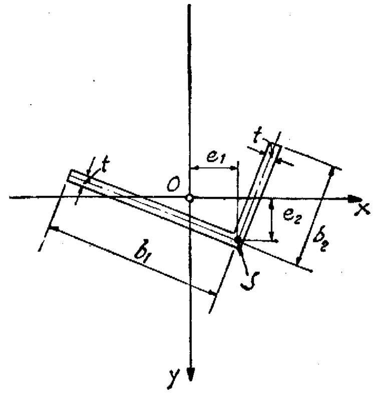


Fig. 48

Angle, Unequal Legs (Fig. 48):

$$\left.\begin{aligned} x_0 &= e_1 \qquad \text{and} \qquad y_0 = e_2 \\ \Gamma &= \frac{t^3}{36}(b_1^3 + b_2^3) \end{aligned}\right\} \tag{226}$$

T-section (Fig. 49):

$$\left.\begin{aligned} x_0 &= 0 \qquad \text{and} \qquad y_0 = e \\ \Gamma &= \frac{t_1^3 b^3}{144} + \frac{t_2^3 d^3}{36} \end{aligned}\right\} \tag{227}$$

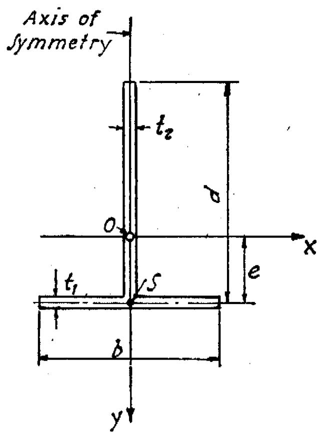


Fig. 49

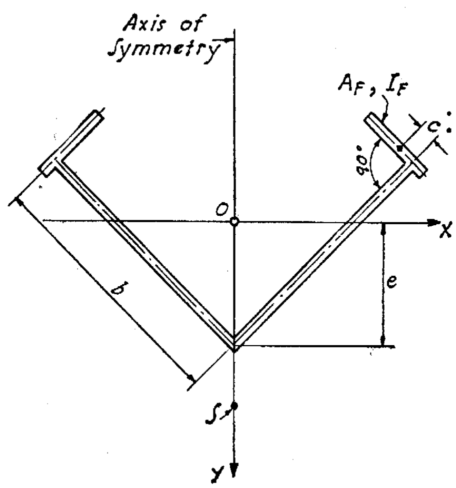


Fig. 50

Section According to Fig. 50:

$$\left.\begin{aligned} x_0 &= 0 \qquad \text{and} \qquad y_0 = \sqrt{2}\left(e + cb^2\frac{A_F}{I_y} - b\frac{I_F}{I_y}\right) \\ \Gamma &= (2e^2 - y_0^2)I_y + 2b(b - 2e)I_F + 4eb^2cA_F \end{aligned}\right\} \tag{228}$$

A_F is the area of each of the flanges, I_F its moment of inertia with reference to the center line of the adjacent leg of the angle, and c the distance of the center of gravity of the flange from this line.

Channel-section (Figs. 51*a* and *b*):

$$\left.\begin{aligned} x_0 = 0 \quad \text{and} \quad y_0 &= -e\left(1 + \frac{d^2A}{4I_y}\right) \\ \Gamma &= \frac{d^2}{4}\left[I_x + e^2A\left(1 - \frac{d^2A}{4I_y}\right)\right] \end{aligned}\right\} \tag{229}$$

where A is the cross-sectional area and I_x, I_y the moments of inertia of the channel. Equations (229) were derived for channels with tapered flanges; these equations are also valid for the more general section shown in Fig. 51*b*. As a limiting case, if the x-axis is an axis of symmetry, Eqs. (229) agree with Eqs. (230) for the symmetrical I-beam.

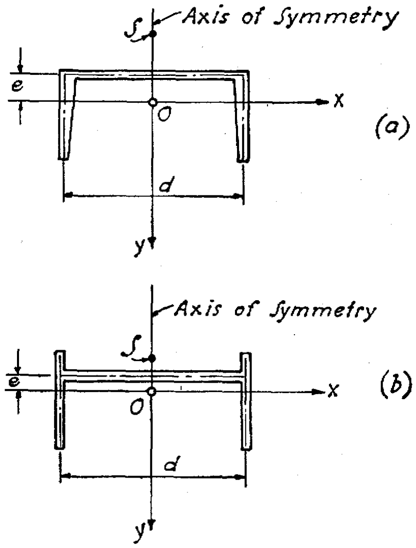


Fig. 51

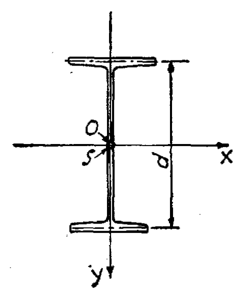


Fig. 52

Symmetrical I-beam (Fig. 52):

$$x_0 = y_0 = 0 \quad \text{and} \quad \Gamma = \frac{d^2 I_y}{4} \tag{230}$$

where I_y is the moment of inertia of the beam with reference to the y-axis.

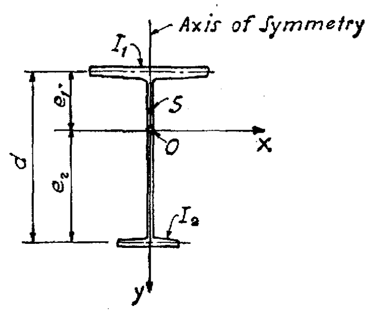


Fig. 53

I-beam with Different Flanges (Fig. 53):

$$\begin{aligned} x_0 &= 0, \\ y_0 &= \frac{e_2 I_2 - e_1 I_1}{I_1 + I_2}, \\ \Gamma &= \frac{d^2 I_1 I_2}{I_1 + I_2} \end{aligned} \tag{231}$$

I_1 and I_2 are the moments of inertia of the upper and lower flange, respectively, with reference to the y-axis. Equations (231) are valid for tapered or otherwise variable flange and web thickness.

Section According to Fig. 54

The thickness of the webs and flanges for this section may be constant or variable without affecting the validity of Eqs. (232).

$$x_0 = 0 \qquad \text{and} \qquad y_0 = -e\left(1 + \frac{b^2A}{4I_y}\right) + 2d\,\frac{I_F}{I_y}$$

$$\Gamma = \frac{b^2}{4}\left[I_x + e^2A\left(1 - \frac{b^2A}{4I_y}\right)\right] + 2d^2I_F - 2bcd^2A_F + b^2deA\,\frac{I_F}{I_y} - 4d^2\,\frac{I_F{}^2}{I_y} \tag{232}$$

A, I_x, I_y are the area and moments of inertia, respectively, of the entire section; A_F and I_F are the area and the moment of inertia of one lower flange with respect to the axis of the web to which it is connected; and c is the distance of the center of gravity of the flange from this axis. The first terms of Eqs. (232) agree with Eqs. (229) for channel-sections.

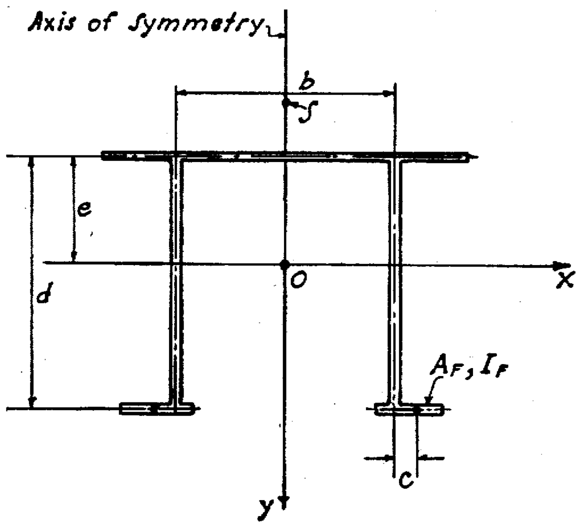


Fig. 54

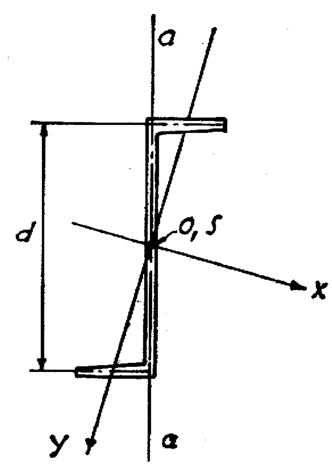


Fig. 55

Z-section (Fig. 55):

$$x_0 = 0, \qquad y_0 = 0, \qquad \Gamma = \frac{d^2}{4}\,I_{a-a} \tag{233}$$

where I_{a-a} is the moment of inertia of the cross section with reference to the center line *a-a* of the web.