# Equilibrium Finite Elements in the Education of Engineers

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Equilibrium is an essential requirement in the design of structural members or mechanical components; without it the design engineer cannot be certain that he/she has placed sufficient material for a design to have the required strength. Yet the standard Conforming Finite Element (CFE) formulation found in most commercial finite element (FE) systems, normally does not offer the reassurance of a strong equilibrium solution unless mesh refinement is conducted.

To illustrate this, consider the parallel-sided rotating disc shown in Figure 1.

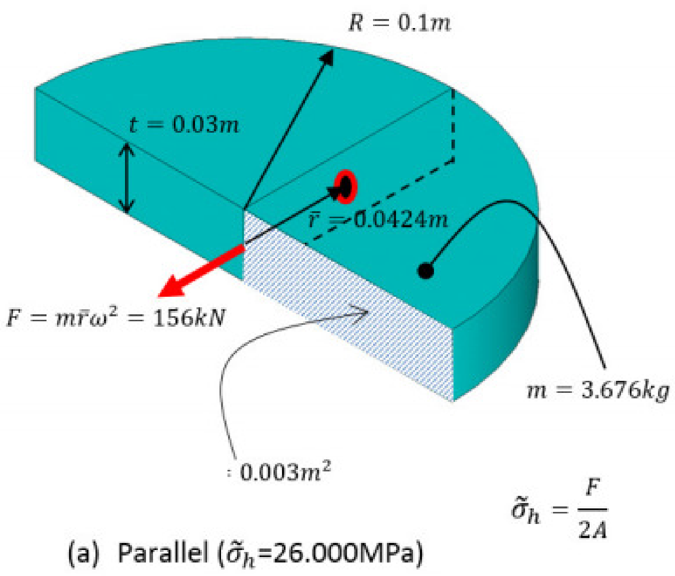


**Figure 1**: Average hoop stress, $\tilde{\sigma}_h$ for parallel-sided rotating disc at $\omega$=1000rad/sec

This disc was studied in [1] where it was shown that the elastic, perfectly-plastic solution involved a uniform hoop stress when the Tresca yield criterion was adopted. This means that the burst speed can be determined exactly by conducting a linear-elastic analysis of the disc taking the average of the parabolically varying hoop stress, c.f., the Lame Equations for rotating discs, and determine by linear scaling the speed at which this average reaches the yield stress for the disc material.

If the disc is modelled using a single four-noded, reduced-integration axisymmetric continuum element then the finite element stress field is more or less constant as shown in Figure 2.

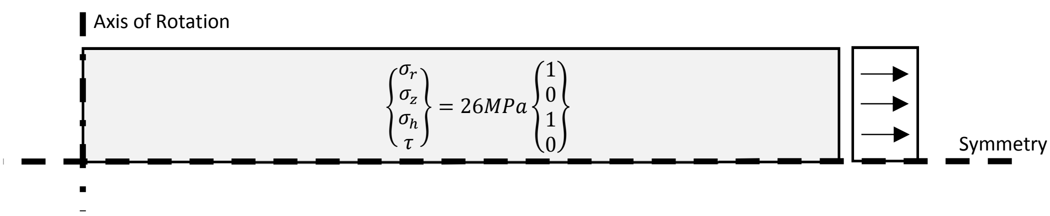


**Figure 2**: Finite element stress field for single element model of disc

The finite element solution clearly violates equilibrium since the constant stress field cannot balance the radial body force field and also has a significant but spurious radial boundary traction on the periphery of the disc. If Figure 2 alone was presented to an engineer he/she would be hard-pressed to recognise the loading that had been applied to the FE model!

The practising engineer should be concerned when seeing examples such as the one just presented since if he/she has come across the plasticity theorems and, in particular, the lower bound theorem then there will be a recognition that the strength of his/her structure/component cannot be guaranteed unless the stresses are in equilibrium with the applied loads.

The weak nature of equilibrium from CFE models has always been a potential issue for caution requiring mesh refinement to resolve. However, with the democratisation of simulation, i.e., placing simulation tools in the hands of ‘persons whose expertise is in other fields’ leads to a potential problem in terms of Simulation Governance. The following definitions were offered by Barna Szabo and have been modified by the author to make them questions, [2]:

**Simulation Governance** – What ***does*** it take~~s~~ for organisations to develop confidence in the results of their commercial simulation projects?

**Democratisation** – How ***does one*** ~~to~~ extend the benefits of numerical simulation to persons whose expertise is in other [engineering] fields?

There is an alternative to the CFE formulation that offers the engineer FE solutions with a strong form of equilibrium irrespective of mesh refinement. Known as the Equilibrium Finite Element (EFE) formulation, this offers the practising engineer protection against a major form of finite element malpractice namely, not conducting sufficient mesh refinement and as such, provides a useful and viable alternative or complement to conventional finite elements (CFE). A theoretical text on EFEs was published last year, [3], and NAFEMS have contracted the authors of this presentation to write a book in their ‘Why Do … ?’ series aimed at the practising engineer, [4].

The presentation will draw upon the authors’ experience both in the education of young engineers and also on their research and development into EFE solutions for practical engineering problems. We will compare and contrast the nature of CFE and EFE solutions to make clear the relative virtues of the two forms of finite element.

**References**

[1] A.C.A. Ramsay, ‘NAFEMS Benchmark Challenge Number 8: *Calculation of the Hoop Burst Speed for Rotating Discs*’, NAFEMS Benchmark Magazine, January 2017.

[2] A.C.A. Ramsay, Review of NAFEMS World Congress 2017, in press.

[3] J.B. Moitinho de Almeida & E.A.W. Maunder, ‘Equilibrium Finite Element Formulations’, Wiley, 2017.

[4] A.C.A. Ramsay & E.A.W. Maunder, ‘Why do Equilibrium Finite Element Analysis?’, NAFEMS, in preparation.