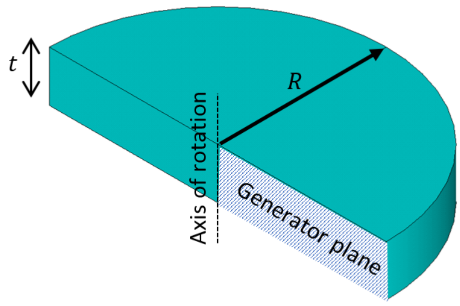
Dimensional Analysis & Numerical Experiments for a Rotating Disc

Most mechanical engineers will have come across dimensional analysis as undergraduates, probably in the field of fluid mechanics. Fewer will have come across it in their practice as engineers particularly if their field of application does not involve fluid mechanics. However, non-dimensional numbers or groups also crop up in relation to structural mechanics. When considering slender chimneys subject to wind induced vibrations, for example, the Strouhal and Scruton numbers become important for determining, respectively, the critical wind velocity and the amplitude of vibration.

The ideas of dimensional analysis are extremely useful in fields other than fluid dynamics and in this technical note the technique is used to understand the relationship between the variables for a solid, parallel-sided rotating disc.

Dimensional Analysis of a Disc

There are five variables for this problem as identified in Figure 1 where the dimensions are provided in terms of mass (*M*), length (*L*) and time (*T*).



Variable	Symbol	Dimension
Radius	R	L
Thickness	t	L
Density	ρ	ML^{-3}
Speed	ω	T^{-1}
Stress	S	MT^{-1}

Figure 1: Parameters for solid, parallel-sided disc – half disc with axisymmetric generator plane

The problem involves five variables and the number of different dimensions for this problem is three thus the number of independent non-dimensional groups is 5-3=2. In order to find these groups, it is useful to form the dimensional matrix for the variables as shown in Figure 2. This matrix collects the indices of the dimensions for each variable with the columns representing the variables and the rows the dimension. Thus, for example, the first column of the dimensional matrix shows that the speed, ω, has dimensions of T^{-1} etc. The two non-dimensional groups have been obtained by inspection but there are more formal methods of matrix analysis that can be adopted to recover the vectors that span the null-space of the dimensional matrix.

$$\begin{matrix} & \omega & \rho & S & R & t \\ T \\ M \\ L \end{matrix}\quad \begin{bmatrix} -1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ -1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} = [0]$$

Dimensional Matrix

Demand/Capacity	Thickness/Radius
$\frac{\rho\omega^2 R^2}{S}$	$\frac{t}{R}$

Figure 2: Formation of two non-dimensional groups

The stresses in a thin (plane stress constitutive model) solid rotating disc are given by the Lamé equations as in Eq(1), where ν is Poisson's ratio.

	Hoop	**Radial**	
Elastic	$\sigma_h = \frac{\rho\omega^2}{8}[(3+\nu)R^2 - (1+3\nu)r^2]$	$\sigma_r = (3+\nu)\frac{\rho\omega^2}{8}(R^2 - r^2)$	(1)

The maximum stresses occur at the centre of the disc (r=0) where the hoop and radial stresses are equal as shown in Eq(2).

Stress at Centre

$$\sigma = \sigma_h = \sigma_r = \frac{\rho\omega^2}{8}(3+\nu)R^2 \quad (2)$$

The average hoop stress over the disc is used later in calculating the plastic limit speed and is given in Eq(3).

Average Hoop Stress

$$\tilde{\sigma}_h = \frac{\rho R^2\omega^2}{3} \quad (3)$$

The Lamé equations consider the stresses in the disc to be bi-axial (two-dimensional) in nature with the third direct stress component, the axial stress, being zero. Considering now the yield of a bi-axial stress system, where the two principal stress are equal, leads to the conclusion that yield is identical whether the yield criterion is Tresca, von Mises or even Maximum Principal Stress. This idea is shown in Figure 3 where the normalisation involves dividing by the yield stress.

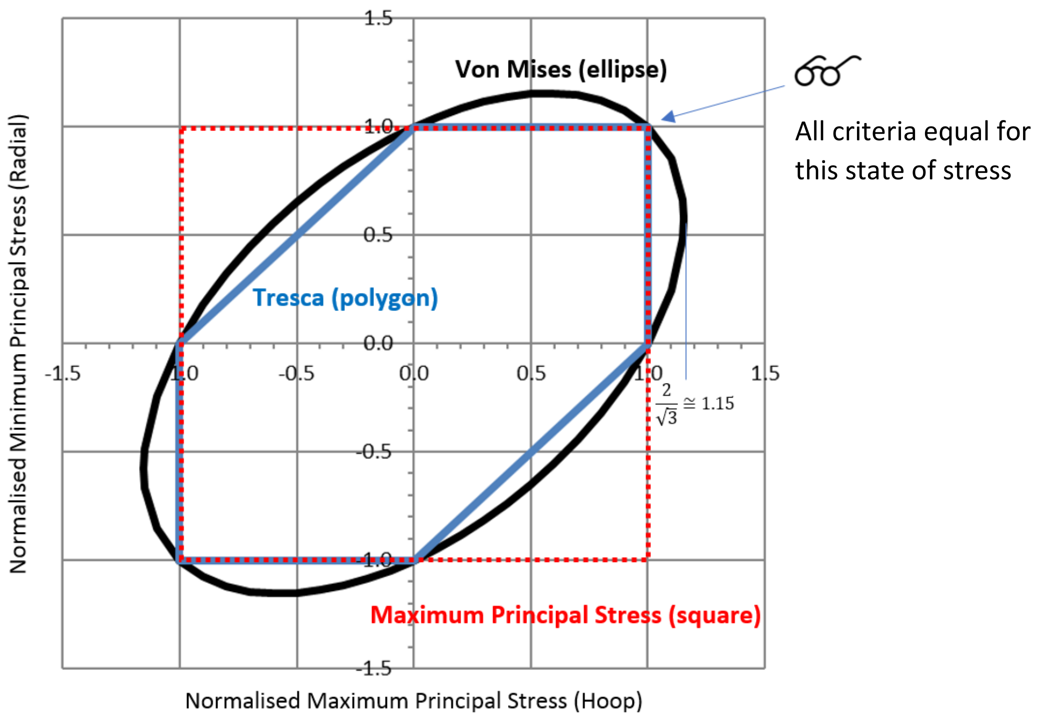


Figure 3: Three yield criterion in normalised principal moment space

Practising engineers will be familiar with the idea that, for a given radius, as the thickness of the disc increases, i.e., increasing the value of the non-dimensional group t/R, the plane stress assumption becomes increasingly invalid with a more appropriate constitutive model being given by a plane strain assumption. They may also realise that the plane strain assumption is only valid when the axial strain is zero, i.e., when the disc is restrained axially on the free faces. If this is not the case, then, as the thickness of the disc increases, the state of stress at the centre will tend towards plane strain whereas the state of stress on the free faces will mimic that of plane stress.

Taking the conditions of plane stress and plane strain (noting that the plane strain constitutive model is obtained by replacing ν with $\nu/(1-\nu)$ in the plane stress model) and substituting the stress at the centre of the disc into the demand/capacity group one obtains the expressions in Table 1; the numerical values are obtained for a Poisson's ratio of 0.3.

	Plane Stress	**Plane Strain**
$\frac{\rho R^2 \omega^2}{S}$	$\frac{8}{3+\nu} = 2.4242'$	$\frac{8(1-\nu)}{3-2\nu} = 2.3333'$

Table 1: Demand/capacity for the two planar cases

The plane stress model assumes a small disc thickness whereas the plane strain model assumes a large thickness with additional constraints on axial displacement and it is seen that the speed to cause first yield for plane strain cases is about 4% lower than that for the plane stress assumption. Although this is a small difference in the sense of general levels of engineering accuracy, it is interesting to speculate how the elastic limit speed varies with disc shape between thin and thick discs particularly when the disc has no axial constraint, i.e., so that the plane strain condition is not met for the entire disc.

Numerical Experiments

With the non-dimensional groups now defined for the solid, parallel-sided disc it is possible to conduct some numerical experiments to develop a more detailed understanding of how the stresses at the centre of the disc are influenced by the shape of the disc. This can be done utilising an axisymmetric finite element model. Such a model will predict the true nature of the stresses in disc as the constitutive relation explicitly involves all components of stress thereby making no assumption about the nature of the axial stress or strain. Using the von Mises stress at the centre of the disc as a stress measure in the demand/capacity group, a series of finite element models for different thickness/radius ratios were analysed and the results are plotted in Figure 4.

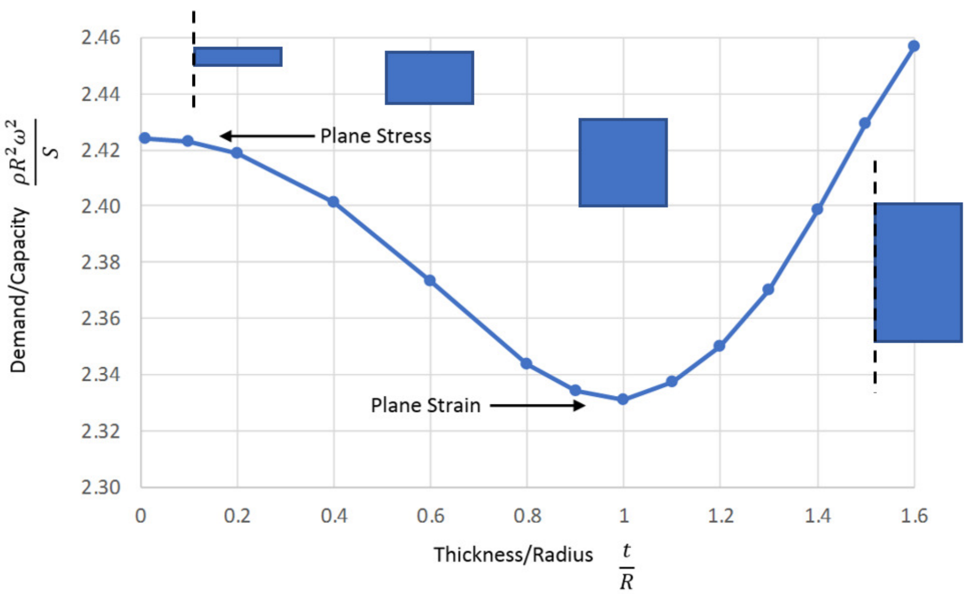


Figure 4: Design/Assessment Chart for disc

The results shown in Figure 4 show:

- For small thickness/radius ratios the theoretical value for plane stress is recovered.

- For thickness/radius ratios in the range 0 to 1, the demand/capacity ratio decreases with the value at unit thickness/radius ratio being close to the theoretical value for plane strain. Remember that the stress being monitored is that at the centre of the disc and even though no axial restraint has been applied to the disc faces, the stress state is tending towards that of plane strain.

- As the thickness/radius ratio increases beyond unity, the demand/capacity ratio also increases.

The point made in the third bullet above needs some further explanation. For small thickness/radius ratios the axial stress is essentially zero but as the thickness/radius ratio increases, the value of the axial stress does likewise. The radial and hoop stresses remain essentially equal to each other. The principal stress system, which began as two-dimensional, becomes increasingly three-dimensional at the centre of the disc. If the axial stress ever reached the value of the radial and hoop stresses, then the stress state would be a hydrostatic state with zero von Mises stress.

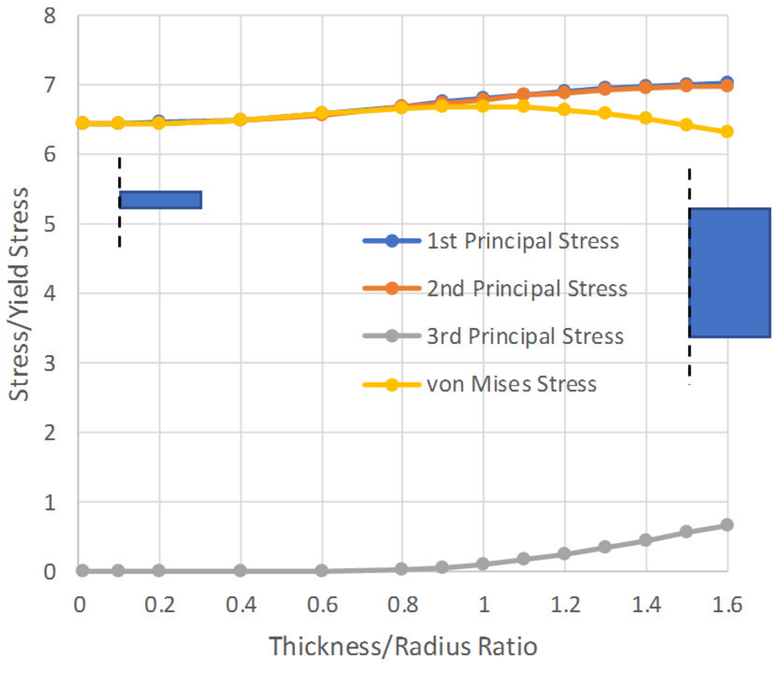


Figure 5: Development of principal and von Mises stress with thickness/radius

Use of the Design/Assessment Chart

The design problem involves seeking a disc thickness or radius for given values of the other four parameters. In contrast, the assessment problem involves determining the elastic limit speed for given values of the other four parameters. The graph presented in Figure 4 can be used as a design/assessment chart as shown in the following two examples.

Assessment Example: An engineer has been asked to establish the speed of failure for a disc. The disc was manufactured from a brittle material having a density of 7800kg/m^3 and a yield stress of 125MPa. It has a radius of 0.5m and a thickness of 0.2m.

Assessment Solution: The thickness/radius ratio of the disc is 0.4 and from the design assessment chart the demand/capacity ratio is about 2.4. For the given material, the failure speed is about 392rad/sec or 3746rev/min.

Design Example: A designer has been asked to determine the thickness of 1m radius disc which is to rotate at 2778rev/min and will be manufactured from a steel with a density of 7800kg/m^3 and a yield stress of 275MPa.

Design Solution: The demand/capacity ratio is about 2.4 for this disc and from the design/assessment chart this offers two limiting solutions with thicknesses of either 0.4m or 1.4m. Discs with thicknesses between these values are unsafe but for thicknesses below 0.4m or greater than 1.4m the design will be safe.

The plastic limit speed, using the Tresca yield criterion, has a theoretical value for solid, parallel-sided discs and this can be established by substituting the average hoop stress into the demand/capacity ratio. This gives a value of precisely three for this non-dimensional group. The ratio between the plastic and elastic limit speeds can then be established and this has a minimum value (in the range of t/R considered here) determined as the square root of 3/2.46 which equals 1.10. The plastic limit speed can, of course, only be realised if the material is sufficiently ductile to allow the appropriate stress redistribution. In the design of turbomachine discs, the ratio of plastic limit speed to design speed should, generally, be greater than 1.20. Thus, on this basis and for the solid disc considered, the design speed is below the elastic limit speed.

Closure

This technical note demonstrates how, through dimensional analysis, the number of variables involved in a problem can be reduced by considering them as dimensionless groups. For the disc, five variables were reduced to two dimensionless groups. The relationship between groups was then determined by numerical simulation, i.e., finite element analysis. The form of the relationship was explained in terms of the concepts of plane stress and plane strain and how the development of the axial stress component, with increasing thickness/radius, leads to a reduction in the von Mises stress at the centre of the disc. It is to be recorded that as thickness/radius increases beyond the range considered here, the position of maximum stress moves away from the centre of the disc, along the centre line towards the face of the disc. The graph of demand/capacity versus thickness/radius provides a design/assessment chart which is extremely simple to use and covers all possible values of the five variables involved. In producing a design/assessment chart such as that in this technical note, it is essential to ensure that the finite element results have been verified. In this case software verification is provided with the close agreement of the demand/capacity ratio for the very thin disc with that from plane stress theory. Meshes of a similar density were used to obtain results for the other thickness/radius ratios and as a final check, solution verification was performed for t/R=1.6. This showed that the results were accurate to within 1%.

Bibliography

Angus Ramsay, **Calculation of the Hoop Burst Speed for Rotating Discs**, NAFEMS Benchmark Challenge Number 8, NAFEMS Benchmark Magazine, April 2017.

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