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BOUNDARY CONDITIONS AND SUB-MODELLING WITH *P*- TYPE HYBRID EQUILIBRIUM PLATE ELEMENTS

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Summary. *Strategies for sub-modelling with hybrid equilibrium plate elements of varying degree are outlined from a practical engineering point of view. Possible types of boundary condition which can be transferred to a sub-domain are considered, and the use of codiffusive tractions is illustrated for a stress concentration problem.*

1 INTRODUCTION

In structural finite element analysis, a relatively coarse global finite element model may be initially designed and analysed in order to capture the main features of structural behaviour. Quantities of interest, such as concentrations of stress-resultants at a particular geometrical feature, may then be obtained with greater accuracy by analysing a refined submodel with boundary conditions transferred from the global model[1]. This submodelling strategy is motivated by various factors, e.g. to simplify complex models into ones with limited numbers of degrees of freedom, and to reduce the potential for ill-conditioning. Other strategies with similar motivations have been proposed, e.g. using substructures in a finite element tearing and interconnecting (FETI) method[2].

Various combinations of finite element models and interface boundary conditions can exist for a submodelling strategy. For example discretisations Ω and Ω_s of the complete domain and a particular subdomain of interest can be of the conforming and/or equilibrating type, with boundary conditions involving displacements and/or tractions transferred along an interface which may or may not coincide with the mesh lines of the global model. The transfer is represented by the map *e* in Equation (1), and is shown diagrammatically in Figure 1,

$$f_{\Omega} \xrightarrow{e} f_{\Omega_s} \tag{1}$$

where *f* denotes excitations, in the form of tractions and/or displacements, applied to the interface. In the case of conventional conforming models *e* can be defined by interpolating displacements from global nodes to those of the submodel, e.g.using master-slave concepts[4].

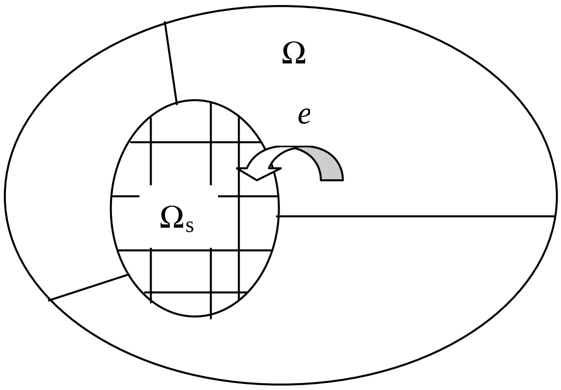


Figure 1: Global model and a submodel.

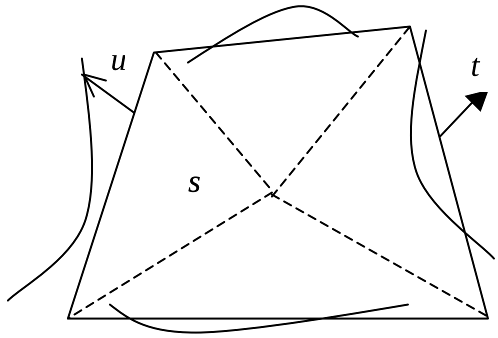


Figure 2: Subdivision of a hybrid element

When Ω_s is formed from hybrid equilibrium elements[5], f can be defined solely by tractions when Ω_s is a subdivision of Ω on the interface and equilibrating tractions are derived from nodal forces[6]. On the other hand when both Ω and Ω_s are formed from hybrid equilibrium elements f can be defined by tractions which maintain strict statical admissibility when Ω_s is a subdivision of Ω on the interface, and by tractions with controllable statical inadmissibilities in the more general case when the interface is formed by an arbitrary "cut" chosen by the engineer. The remainder of this paper considers only the use of hybrid equilibrium elements in the context of plate problems.

2 HYBRID EQUILIBRIUM ELEMENTS

The hybrid type of equilibrium element is illustrated in Figure 2. It has an inherent subdivision into primitive triangular subdomains so as to eliminate or control the existence of spurious kinematic modes. Internal stress fields s of degree p are defined to be statically admissible though semi-continuous. A basis for edge displacement fields u is defined independently for each edge as complete Legendre polynomials up to degree p. The dual basis for edge tractions t is defined by the same polynomials with appropriate scaling, i.e.

$$\boldsymbol{u} = \boldsymbol{V}\boldsymbol{v}\,;\; \boldsymbol{t} = \boldsymbol{G}\boldsymbol{g}\,;\ \text{where}\ \boldsymbol{G} = \boldsymbol{V}.\boldsymbol{S}\ \text{and}\ \boldsymbol{S} = \left[\int_{edge} \boldsymbol{V}^{T}\boldsymbol{V}\,d\boldsymbol{e}\right]^{-1} \tag{2}$$

The discretisations of the interface will be assumed to involve straight edges, whilst the external boundary of Ω may include curved edges in a similar way as for isoparametric conforming elements.

3 SUBDOMAIN BOUNDARY CONDITIONS

On the edges of a subdomain, tractions are applied as modes $\boldsymbol{g}$ defined by Equation (3).

$$\boldsymbol{g} = \int_{edge} \boldsymbol{V}^{T}\tilde{\boldsymbol{t}}\,d\boldsymbol{e} \tag{3}$$

where $\tilde{\boldsymbol{t}}$ denotes tractions from Ω. Strong equilibrium is maintained with co-diffusive tractions when Ω_S subdivides Ω on the interface and the local degree of $\Omega_S \geq$ degree of Ω. Relaxation of co-diffusivity may be considered when (a) it is sufficient, by appeal to the principle of St Venant, to transfer only the basic modes of edge traction which represent the resultants; or (b) Ω_S doesn't subdivide Ω, e.g. when the interface is formed by an arbitrary cut through elements of Ω. In this case discontinuities are to be expected in $\tilde{\boldsymbol{t}}$ along an edge of Ω_S.

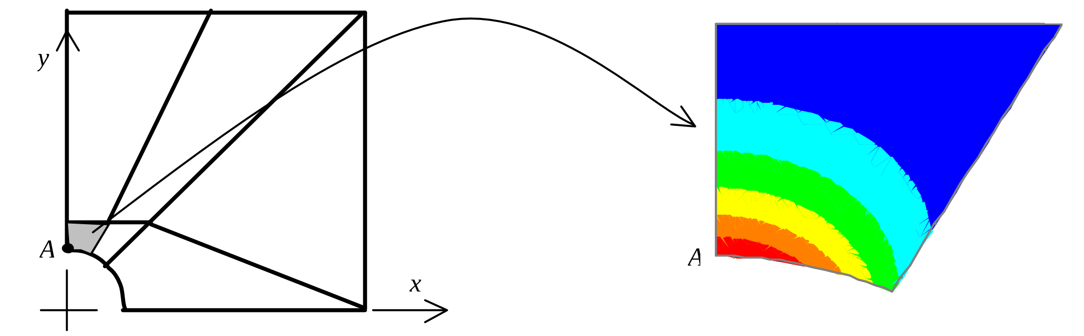
4 NUMERICAL EXAMPLE

4.1 A classical problem with practical significance

A classical problem appropriate for the study of sub-modelling is that of the plate-membrane with a circular hole. The hole concentrates the stress and the aim is to obtain an accurate prediction of this peak stress. This problem characterises much of the analysis conducted in the field of practical mechanical engineering where, typically, such peak stresses limit the fatigue life, high and/or low-cycle, of a component. If modelled correctly this problem has an analytical solution with which the FE results may be compared. Correct modelling requires that the boundary conditions be derived from the analytical stress field for the particular geometry of the FE model. The analytical stress field is given as:

$$
\begin{aligned}
\sigma_x &= \sigma_\infty\{1 - \frac{\boldsymbol{a}^2}{\boldsymbol{r}^2}(\frac{3}{2}\cos 2\theta + \cos 4\theta) + \frac{3}{2}\frac{\boldsymbol{a}^4}{\boldsymbol{r}^4}\cos 4\theta\} \\
\sigma_y &= \sigma_\infty\{0 - \frac{\boldsymbol{a}^2}{\boldsymbol{r}^2}(\frac{1}{2}\cos 2\theta - \cos 4\theta) - \frac{3}{2}\frac{\boldsymbol{a}^4}{\boldsymbol{r}^4}\cos 4\theta\} \\
\tau_{xy} &= \sigma_\infty\{0 - \frac{\boldsymbol{a}^2}{\boldsymbol{r}^2}(\frac{1}{2}\sin 2\theta + \sin 4\theta) + \frac{3}{2}\frac{\boldsymbol{a}^4}{\boldsymbol{r}^4}\sin 4\theta\}
\end{aligned} \tag{4}
$$

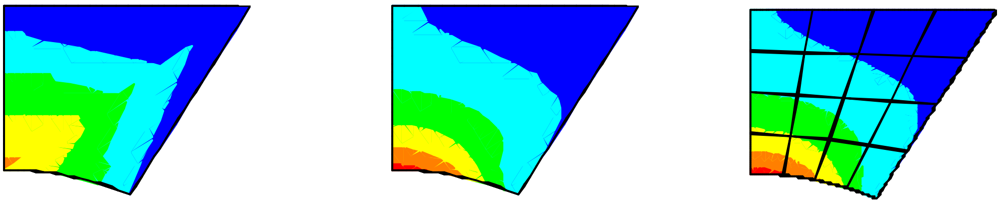
where r and θ are polar position ordinates, a is the hole radius and σ_∞ is the (uniform) value of σ_x at $r = \infty$. The FE model used for this problem utilises symmetry by modelling only a



(a) Domain and mesh Ω (b) Analytical stress concentration factor = 3.0

Figure 2: Finite element model and analytical stress s_x field in a subdomain

quarter of the plate which is designated as the domain and the FE mesh Ω is shown in Figure 2(a). Ω consists of six elements with a biasing towards point A where the stress concentration factor (scf) is sought. The curved boundary is modelled by three piecewise quadratic edges defined with end and midpoints on the circular arc. Convergence of the scf is illustrated in Figure 3 when Ω_s is a refinement of the single element of Ω shown in Figure 2.



$p_s = 1 \Rightarrow$ scf = 2.57 $p_s = 4 \Rightarrow$ scf = 2.85 $p_s = 2$, $h = 2 \Rightarrow$ scf = 2.84

Figure 3: Convergence of the scf for s_x, p_s is the degree, h is the level of h-refinement

5 CONCLUSIONS

- Submodelling strategies have been outlined for use with *p*-type hybrid equilibrium elements in the context of modeling plates in which boundary conditions are transferred as tractions.
- Boundary tractions may be codiffusive and lead to the preservation of statically admissible solutions within a subdomain. However inherent errors in these tractions imply that solutions will not necessarily converge to correct values of quantities of interest.
- Considerable freedom is available to the engineer in specifying an interface for a subdomain and its discretisation. Various options present themselves for the relaxation of the use of strictly codiffusive tractions, and these need to be investigated further for their effectiveness.

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