A Basis for an Axisymmetric Hybrid-Stress Element

E.A.W. Maunder*

University of Exeter,
Department of Engineering, School of Engineering and Computer Science,
North Park Road, Exeter, EX4 4QF, England.
e-mail:e.a.w.maunder@exeter.ac.uk

A.C.A. Ramsay

P.C.A. Engineers Limited,
Homer House, Sibthorp Street, Lincoln, LN5 7SB, England.
e-mail:angus_ramsay@yahoo.co.uk

Abstract

Recent developments of hybrid-stress elements are presented suitable for modelling axisymmetric problems when a strong form of equilibrium is required. Potential instability problems due to spurious kinematic modes are shown to be avoided by appropriate selection of statically admissible stress fields and edge displacements, and by the use of the macro-element concept. This paper is restricted to elements for problems where a hole occurs along the axis of symmetry; simple numerical examples are included to illustrate element characteristics, to verify the associated software, and to compare with conventional conforming displacement models. Suggestions are included for future work to develop axisymmetric hybrid models further.

1 Introduction

Finite element models formed from hybrid-stress (HS) equilibrium elements produce statically admissible (SA) stress fields which enable, *inter alia*, safe lower-bound solutions to be achieved in the limit analysis of problems in continuum mechanics. This property is not shared with models based on conventional displacement elements, or normally with models involving the hybrid elements based on internal SA fields and frame functions for boundary displacements [1]. The formulation presented in this paper is based on later hybrid concepts which rely on independent edge displacement fields to enforce a stronger form of equilibrium [2].

The robust formulation of an HS element requires elimination or control of any spurious kinematic modes (skm's) that may exist. For two- and three-dimensional continua and for plate-bending problems this has been achieved by assembling *macro* elements from groups of triangular *primitive* elements – see [3-5] for example. Whilst skm's may still exist in the interior of such an assembly, they do not affect the exterior edges of the macro and, therefore, do not lead to inadmissible configurations of boundary tractions which can occur in the presence of skm's. In order to extend the range of available HS elements, and because of its practical significance in the pressure vessel and turbo-machinery industries, an axisymmetric element has been developed and this forms the subject of the present paper.

The paper proceeds in Section 2 to review the hybrid-stress formulation in the context of axisymmetric problems before detailing a primitive element based on rational stress fields in Section 3. The number and nature of skm's for a range of SA stress fields is established for the triangular primitive, and this is followed in Section 4 by consideration of a triangular macro-element. It is there demonstrated that skm's are suppressed and some simple examples are presented. Section 5 contains conclusions and recommendations for future work.

2 Hybrid formulation

In the context of axisymmetric problems, hybrid stress fields require four components which must be SA. These are represented in Equation (1) where $\{s\}$ denotes the independent stress parameters.

$$\{s\} = \begin{Bmatrix} s_r \\ s_z \\ s_q \\ t \end{Bmatrix} = [S]\{s\} \tag{1}$$

In the absence of body forces, the local differential equations of equilibrium are given by Equations (2) and (3). Equilibrium in the direction of the axis of symmetry z requires:

$$\begin{aligned} & r\left(\frac{\partial s_z}{\partial z} + \frac{\partial t}{\partial r}\right) + t = 0 \quad \text{or} \\ & r\frac{\partial s_z}{\partial z} + \frac{\partial (rt)}{\partial r} = 0 \end{aligned} \tag{2}$$

Equilibrium in the radial direction requires:

$$\begin{aligned} & r\left(\frac{\partial s_r}{\partial r} + \frac{\partial t}{\partial z}\right) + s_r = s_q \quad \text{or} \\ & \frac{\partial (rs_r)}{\partial r} + r\frac{\partial t}{\partial z} = s_q \end{aligned} \tag{3}$$

The stress component s_q is then dependent on s_r and t.

Hybrid displacement fields $\{d\}$ are defined along each edge of an element in Equation (4) where $\{v\}$ denotes the independent edge displacement parameters.

$$\{d\} = [V]\{v\} \tag{4}$$

Edge tractions $\{t\}$ are defined as forces per unit length, and these equilibrate with internal stresses according to Equation (5).

$$\{t\} = h\begin{Bmatrix} s_n \\ s_t \end{Bmatrix} = r \cdot dq[\bar{S}]\{s\} \text{ where } [\bar{S}] = [T][S] \tag{5}$$

In Equation (5) an element is considered as a wedge with thickness proportional to radius r, i.e. thickness $h = r \times dq$ when the wedge subtends an angle dq at the axis of symmetry. s_n and s_t are the normal and tangential components of stress at a point on the edge of an element which are obtained from the four internal stress components by use of the transformation matrix [T].

Then the work done on the boundary of an element can be expressed by Equation (6), where t in dt denotes the tangential coordinate on the boundary.

$$\oint_{\partial W} \{d\}^T \{t\} dt = \{v\}^T \left[\oint_{\partial W} [V]^T [\bar{S}] r dq dt \right] \{s\} = \{v\}^T [D] \{s\} = \{s\}^T [D]^T \{v\} \tag{6}$$

Without loss of generality it is assumed that dq = 1, and then [D] and equilibrating edge traction modes {g} dual to edge displacement modes {v} are defined by Equation (7).

$$[D] = \oint_{\partial W} [V]^T [\bar{S}] r dt \quad \text{and} \quad \{g\} = [D]\{s\} \tag{7}$$

The nullspace of [D] contains the hyperstatic modes i.e. internal stresses with zero tractions. Generalised deformations {e} dual to {s} are defined by Equation (8).

$$\{e\} = [D]^T \{v\} \tag{8}$$

so that the boundary work in Equation (6) is also given by $\{s\}^T\{e\}$. The nullspace of $[D]^T$ contains the kinematic modes i.e. modes of edge displacements which do no work with any of the stress fields – including rigid body and spurious kinematic modes.

The natural flexibility matrix for an element is defined in Equation (9).

$$[F] = \iint [S]^T [f][S] r dr dz \quad \text{where } \{e\} = [f]\{s\} \tag{9}$$

When general tractions $\{\bar{t}\}$ are prescribed for an edge G_t , they are represented by the traction vector $\{g_{G_t}\}$, dual to $\{v_{G_t}\}$, defined by Equation (10).

$$\{g_{G_t}\} = \int_{G_t} [V]^T \{\bar{t}\} dt \tag{10}$$

Compatibility and equilibrium equations for a hybrid element excited by edge tractions (in the absence of body forces or initial strains) are collected in Equation (11).

$$\begin{bmatrix} -F & D^T \\ D & 0 \end{bmatrix} \begin{Bmatrix} s \\ v \end{Bmatrix} = \begin{Bmatrix} 0 \\ g \end{Bmatrix} \tag{11}$$

where {g} represents the total prescribed tractions typified by the edge components in Equation (10).

3 Triangular primitive hybrid elements

3.1 A family of elements based on rational polynomials for stress fields

Stress fields are defined as rational polynomials in the form

$$s = \frac{1}{r} \cdot f(z) + g(r,z) \tag{12}$$

where f and g are polynomials of degree (p + 1) and p respectively.

The rationale behind this choice of stress fields is as follows: if polynomials are limited to the form $g(r,z)$ as in the case of membrane or planar stress equilibrium models [4], then spurious kinematic modes exist both for primitive and for macro-elements. For example, when $p = 0$ for both stress and displacement fields, there are 3 spurious kinematic modes for the primitive and the triangular macro which involve the external edges. When $p = 1$ for both fields there are 5 and 3 spurious kinematic modes for the primitive and the macro respectively. These numbers will only tend to increase if the edge displacements are increased in degree to (p + 1) in order to enforce co-diffusivity of tractions. As will be demonstrated in Section 4, the inclusion of the rational polynomial $f(z)/r$ in the stress fields, and polynomials of degree (p + 1) for the edge displacement fields removes spurious kinematic modes from the macros, at least for elements of degree ≤ 3.

It should however be noted that if an element touches the axis of symmetry, then r = 0 at such contact points. In this case the stress fields become singular, and the natural flexibility matrix involves strongly singular integrals. Alternative formulations for elements in such positions may involve (a) excluding the rational terms and attempting to arrange the mesh geometry in such a way as to prevent spurious kinematic modes from propagating through the mesh; or (b) incorporating simplifying concepts from hybrid stress boundary element methods [6]. These alternatives will be considered in a future paper.

3.1.1 A basis for statically admissible stresses.

According to Equation (12) rational polynomial stress fields can be expressed as $\sum a_{m,n} r^m z^n$ with limits on the exponents $(m+n) \le p,\ -1 \le m \le p,\ 0 \le n \le p$, e.g.

$$s_r = a_{-1,0}\frac{1}{r} + a_{-1,1}\frac{z}{r} + a_{-1,2}\frac{z^2}{r} + \cdots + a_{-1,p+1}\frac{z^{p+1}}{r} + a_{0,0} + a_{1,0}r + a_{0,1}z + \cdots + a_{0,p}z^p \tag{13}$$

or $s_r = \frac{1}{r} f_r(z) + g_r(r,z)$.

s_r contains (p + 2) + 0.5(p + 1)(p + 2) = 0.5(p + 2)(p + 3) terms. Similar expressions are used for components s_z, t, and s_q with coefficients a replaced by b, c and d respectively leading to 2(p + 2)(p + 3) independent stress fields. The total number of independent SA stress fields is then reduced since (i) s_q is dependent on the other three components according to equilibrium Equation (3), and (ii) equilibrium Equation (2) leads to constraints which are developed as follows:

$$\frac{\partial(rt)}{\partial r} = c_{0,0} + 2c_{1,0}r + c_{0,1}z + 3c_{2,0}r^2 + 2c_{1,1}rz + c_{0,2}z^2 + \cdots + c_{0,p}z^p$$

$$r\frac{\partial s_z}{\partial z} = b_{-1,1} + 2b_{-1,2}z + \cdots + (p+1)b_{-1,p+1}z^p + b_{0,1}r + b_{1,1}r^2 + 2b_{0,2}rz + \cdots + pb_{0,p}rz^{p-1}$$

The coefficients must be related by the 0.5(p + 1)(p + 2) equations:

$$\begin{aligned} &b_{-1,1} + c_{0,0} = 0 \\ &b_{0,1} + 2c_{1,0} = 0 \\ &2b_{-1,2} + c_{0,1} = 0 \\ &\vdots \\ &(p+1)b_{-1,p+1} + c_{0,p} = 0 \end{aligned} \tag{14}$$

This leaves the total number of independent SA stress fields to be defined by Equation (15).

$$n_s = 2(p+2)(p+3) - 0.5(p+2)(p+3) - 0.5(p+1)(p+2) = (p+2)(p+4) \tag{15}$$

Independent stress components may be selected to correspond to the coefficients of s_r , s_z and the coefficients of t in the function f_t, and then the coefficients of g_t and s_q become dependent. It should be noted from equilibrium Equation (3) that:

$$s_q = \frac{\partial(rg_r)}{\partial r} + \frac{\partial f_t}{\partial z} + r\frac{\partial g_t}{\partial z} \tag{16}$$

which is a simple polynomial form which excludes all rational terms.

3.1.2 A basis for edge displacements

Edge displacements are defined by complete polynomials of degree (p + 1) to ensure co-diffusivity of tractions in the form (rs).

Axisymmetric problems of disks with central holes loaded by uniform radial pressures create radial and circumferential stress fields involving $1/r^2$ terms as defined in the Lamé equations [7,8]. Thus it is tempting to include $f(z)/r^2$ terms in the stress field polynomials, however this implies the need for rational displacement and traction functions for edges, and the present paper restricts attention to the simpler polynomial forms.

3.2 Spurious kinematic and hyperstatic modes

The spurious kinematic and hyperstatic modes are contained in or form the nullspaces of $[D]^T$ and $[D]$ respectively as stated in Section 2. As has been noted in [9], the numbers of these modes cannot be determined simply from counting the numbers of independent stress and displacement fields in a hybrid element; knowledge of rank[D] is also necessary. This has been determined numerically by singular value decomposition of [D] for $0 \le p \le 3$, and the dimensions n_s, n_{hyp}, n_v, and n_{skm} of the vector spaces corresponding to the statically admissible stress fields, the hyperstatic stress fields, edge displacements, and the spurious kinematic modes are presented in Table 1. These dimensions are related by Equation (17) where n_{rbm} denotes the number of independent rigid body modes. For the axisymmetric element there is only one such mode, and this involves translation in the *z* direction.

$$n_{skm} = n_v - n_s - n_{rbm} + n_{hyp} \tag{17}$$

p	n_s	n_{hyp}	n_v	n_{skm}	n_{rbm}
0	8	0	12	3	1
1	15	1	18	3	1
2	24	4	24	3	1
3	35	9	30	3	1

Table 1: Dimensions of stress and displacement spaces.

A single hyperstatic mode exists when $p = 1$, and this is illustrated in Figure 1 with the aid of a family of stress trajectories. The individual stress components for this particular element are given in Equation (18).

$$\begin{Bmatrix} s_r \\ s_z \\ s_q \\ t \end{Bmatrix} = \begin{Bmatrix} \frac{4}{r} - 6 + 2r \\ \frac{8}{r} - \frac{z^2}{r} - 10 + 3r \\ -8 + 6r \\ -\frac{2z}{r} + 2z \end{Bmatrix} \tag{18}$$

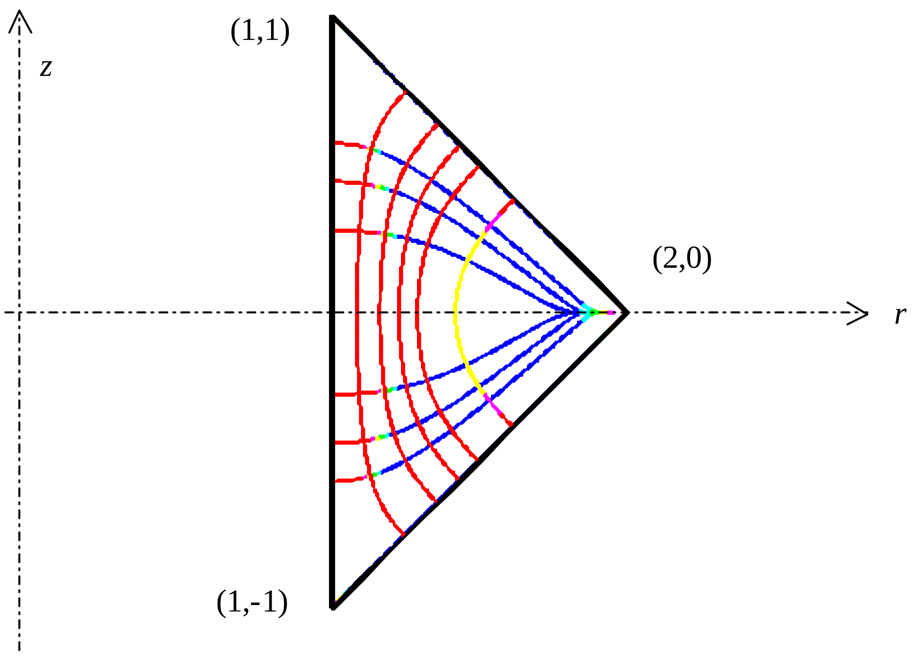

Figure 1: Stress trajectories for the hyperstatic stress field when $p = 1$.

4 Macro-elements

The experience with membrane and plate elements [4-6] indicates that the malignant spurious kinematic modes which produce an unstable finite element model may be suppressed by combining primitive elements into macro-elements e.g. triangular macro-elements containing three primitive triangular elements.

4.1 Triangular macro-elements

The general arrangement of a macro-element is illustrated in Figure2.

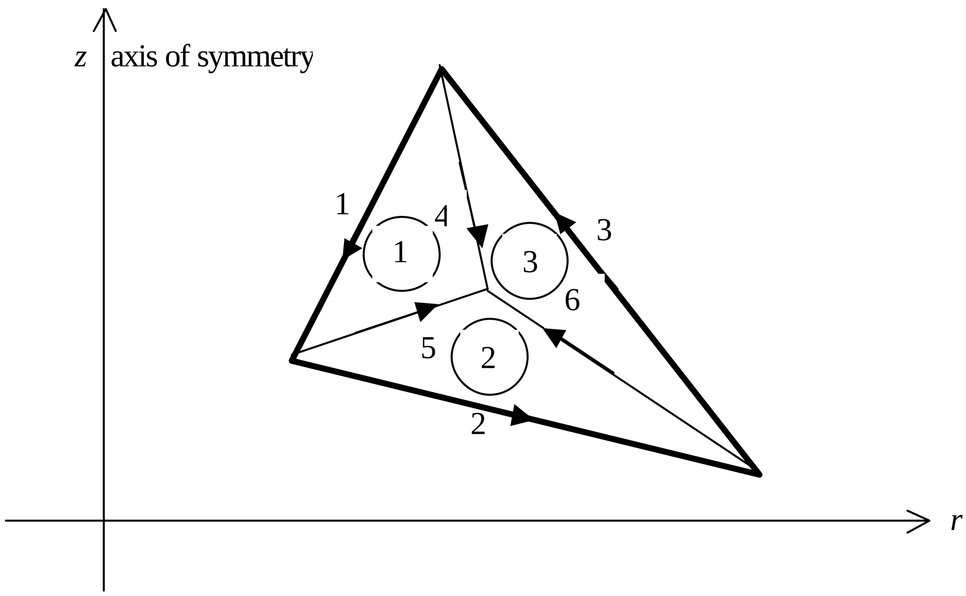

Figure 2: General arrangement of a macro-element.

The macro-element has three external and three internal edges, and its stability is investigated by determining the rank of the matrix [D] assembled for the macro. This has been done numerically for a macro having corner coordinates (1,1), (3,1.5), and (2,3), and an internal node at (2,2). This procedure confirms that the macro is **completely free of spurious kinematic modes** for the same values of p considered in Table 1. The number of hyperstatic modes for the macro is given in Table 2.

Degree p	0	1	2	3
n_{hyp}	13	22	37	58

Table 2: Number of hyperstatic stress fields for the triangular macro-element.

4.2 An illustrative problem

Some simple example problems are included in this section based on the single macro-element in Figure 3. The first three examples are intended to confirm the feasibility of the hybrid element, and to verify the program that has been developed for this macro-element.

To these ends the macro-element with p = 0 is loaded by edge tractions which equilibrate with the Trefftz type stress fields contained within the stress space, i.e.

$$\{s\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \text{ (case 1), } \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \text{ (case 2), and } \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1/r \end{Bmatrix} \text{ (case 3).} \tag{19}$$

Trefftz type stress fields are not only SA, but also the corresponding strains are compatible. The implications of this are that the recovered stress fields should be correct, and the recovered edge displacements should also be correct for the modes included by the complete polynomials. The correct stress fields were recovered in each case. In cases 1 and 2 the strain fields are constant and the correct displacement fields are linear, and are defined to within a rigid body mode in Equations (20) and (21).

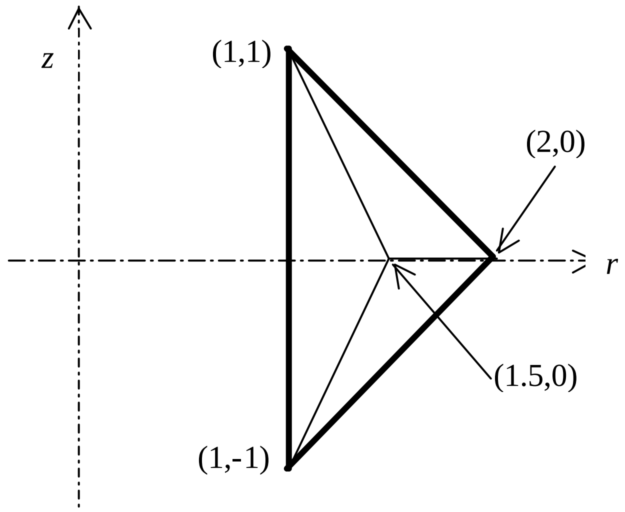

Figure 3: Macro-element for the example problems.

$$\begin{Bmatrix} u \\ w \end{Bmatrix} = \frac{1}{E} \begin{Bmatrix} -nr \\ z \end{Bmatrix} \tag{20}$$

$$\begin{Bmatrix} u \\ w \end{Bmatrix} = \frac{1}{E} \begin{Bmatrix} (1-n)r \\ -2nz \end{Bmatrix} \tag{21}$$

The recovered edge displacements conform with these displacement fields, and they are illustrated in Figure 4.

However in case 3 the linear edge displacements for the hybrid element cannot conform with the correct displacement field, which is given in Equation (22) to within a rigid body displacement in the *z* direction.

$$\begin{Bmatrix} u \\ w \end{Bmatrix} = \frac{1}{E} \begin{Bmatrix} 0 \\ 2(1+n)\log_e r \end{Bmatrix} \tag{22}$$

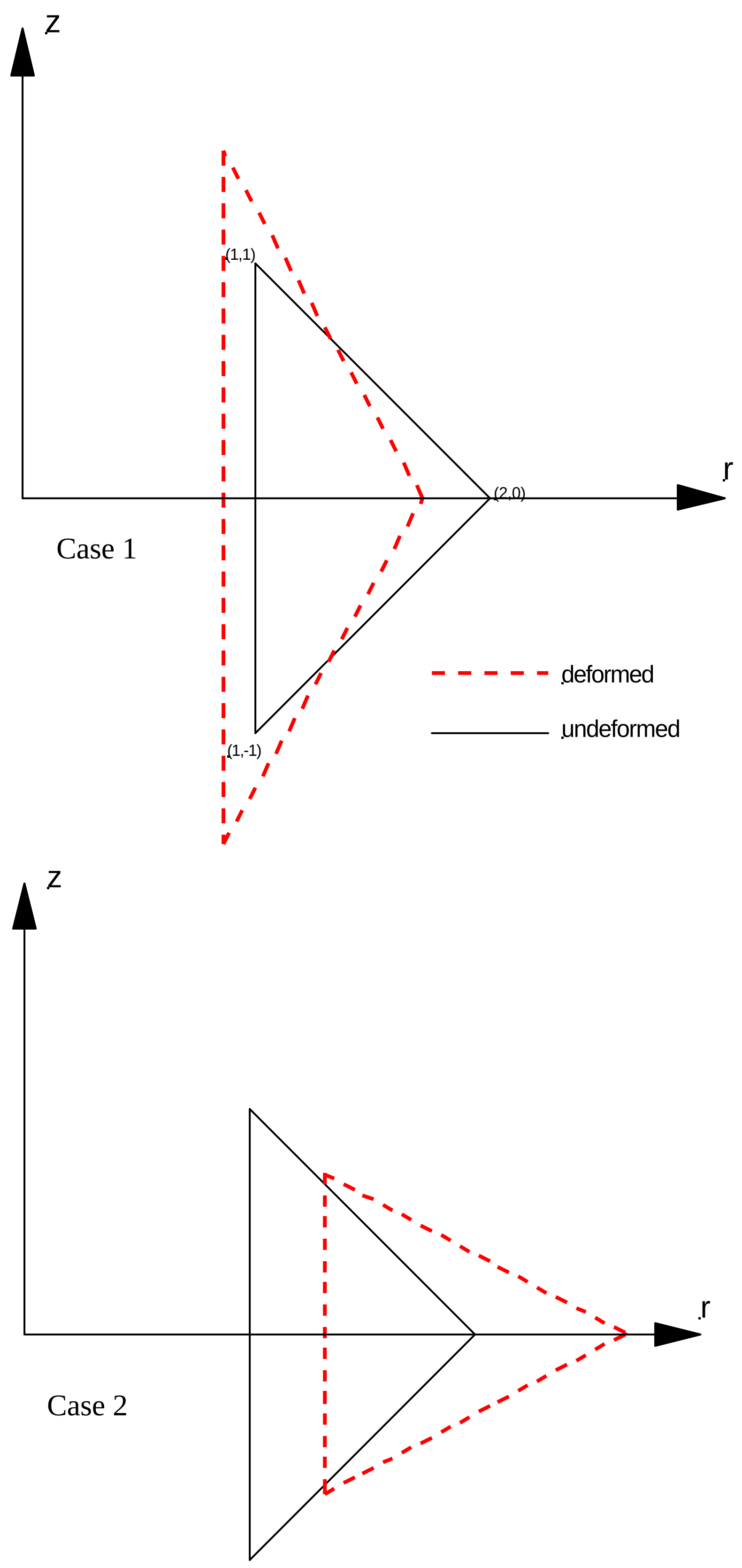

Figure 4: Edge displacements for cases 1 and 2.

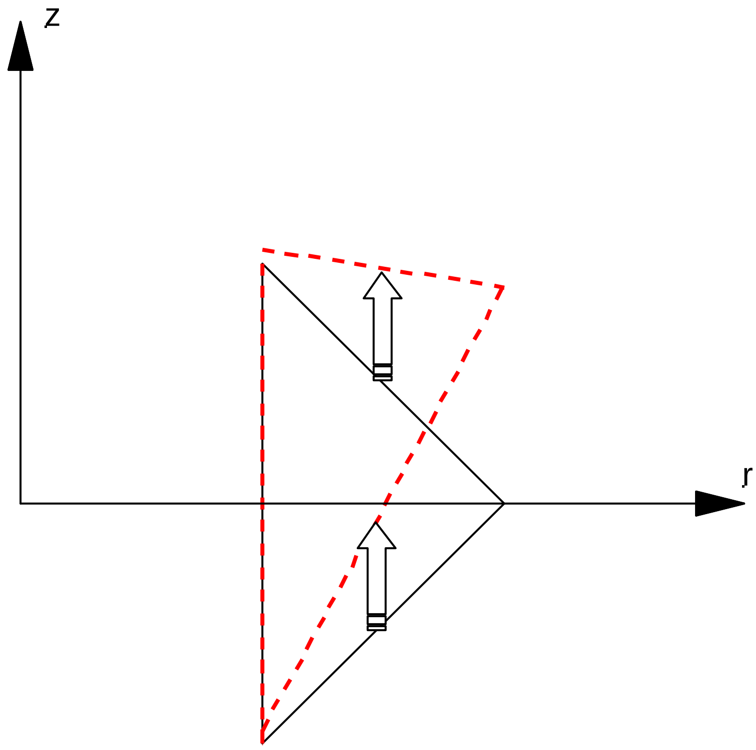

Figure 5: Edge displacements for case 3 for the hybrid macro-element.

The edge displacements for case 3 are illustrated in Figures 5 and 6. Although there are gaps at two of the corners of the macro it should be noted that the edge displacements are correct in so far as the constant and linear modes of displacement are concerned. This is to be expected when the natural flexibility matrix is formed by very accurate integration schemes. In the present case 10 point Gauss integration was used for each edge after using Green's theorem to transform area integrals to boundary integrals.

The fourth example loads the same macro-element with a pressure of 100 units on the edge with $r = 1$, and with $E = 210$, $n = 0.3$. This example simulates a torus with a triangular cross-section subjected to an internal pressure. Results for strain energy for the torus were obtained for p in the range 0 to 3, and for a displacement model based on the same geometrical arrangement of primitive elements but with the degree of the displacement field in the range 2 to 5, and also for a reference solution based on displacement model with a very fine mesh. Results are given in Table 3.

p	Hybrid macro-element Strain energy	Displacement model Strain energy
0	1.198	
1	1.109	
2	1.078	0.995
3	1.070	1.037
4		1.055
5		1.062
Reference solution		1.0679

Table 3: Strain energies of equilibrium and conforming models scaled by 10^6.

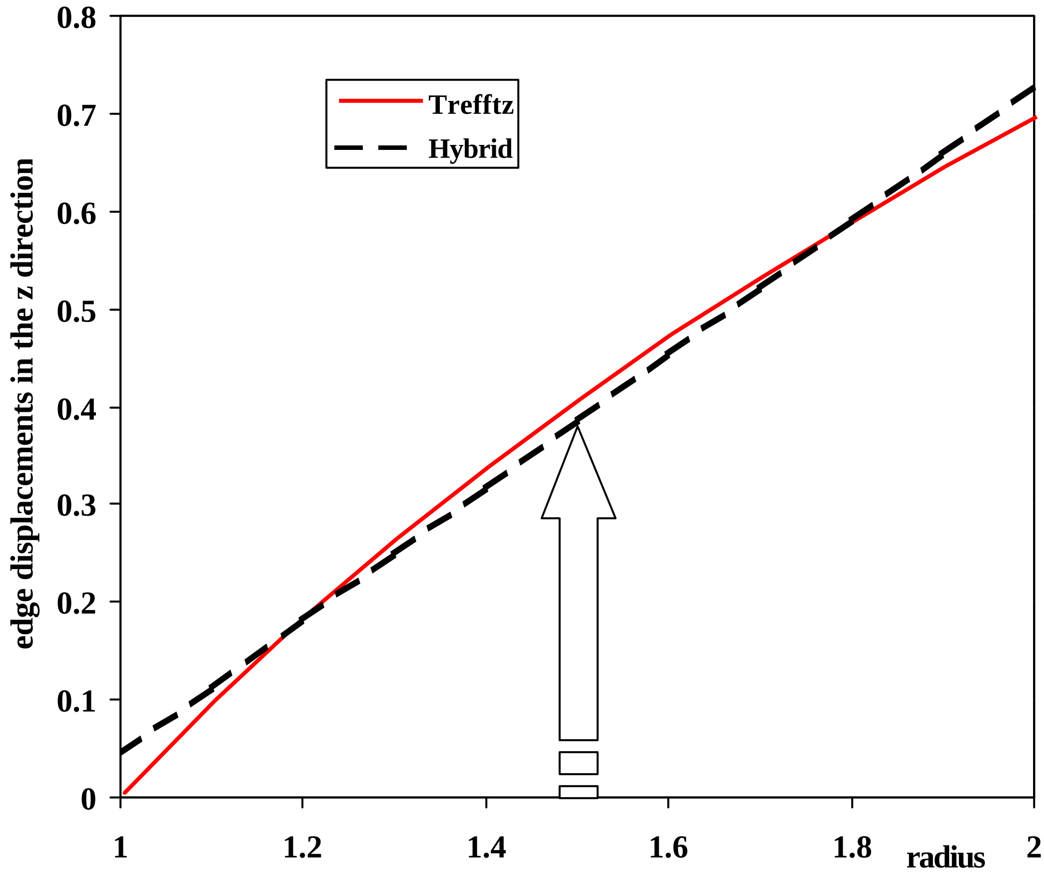

Figure 6: Edge displacements from Trefftz and hybrid element solutions scaled by the shear modulus.

5 Conclusions

- An equilibrium hybrid macro-element in triangular form is feasible for axisymmetric models without spurious kinematic modes when the stress fields are based on rational polynomials. The absence of such modes has been demonstrated for polynomials of low degree. Further work is intended to consider the stability of triangular and quadrilateral macro-elements for general higher degree formulations, and when r^2 is included in the denominator of the stress fields.

- Initial comparisons with conforming displacement models indicate that, for linear elastic behaviour, better quality stress fields can be obtained from equilibrium models of similar degree.

- The present paper has considered elements for which $r > 0$ at all points. Further work is necessary to include elements which touch the axis of symmetry and where the use of rational polynomials may lead to singularities.

- Axisymmetric elements in general tend to lead to ill-conditioned equations when their size is reduced compared with their average radial coordinates [10]. Initial investigations with the present equilibrium elements confirm this trend especially when the degree is increased. Further work is intended to investigate the potential limitations caused by this effect, and how the formulation may be amended so as to improve conditioning.

- Further work is also proposed to include other forms of excitation, e.g. inertia body forces and thermal effects which are of particular concern in the analysis of turbo-machinery.

References

[1] T.H.H.Pian. Evolution of assumed stress hybrid finite element. In: J. Robinson, ed., *Accuracy, Reliability and Training in FEM Technology*, 602-618. Robinson & Associates, Wimborne, 1984.

[2] J.P.M. Almeida, J.A.T. Freitas. Alternative approaches to the formulation of hybrid equilibrium finite elements. *Computers & Structures*, **40**: 1043-1047, 1991.

[3] E.A.W. Maunder, A.C.A. Ramsay. Quadratic Equilibrium Elements, in: J. Robinson, ed., *FEM Today and the Future,* 401-407, Robinson and Associates, Wimborne, 1993.

[4] E.A.W. Maunder, J.P.M. Almeida, A.C.A. Ramsay. A general formulation of equilibrium macro-element with control of spurious kinematic modes. *Int. J. Num. Meth. Eng*, **39**: 3175-3194, 1996.

[5] E.A.W. Maunder. Hybrid elements in the modelling of plates, in B.H.V.Topping, ed., *Finite Elements: Techniques and Developments*, Civil-Comp Press, 165-172, 2000.

[6] N.A. Dumont. The Hybrid Element Method. In: C.A. Brebbia, W. Wendland, G. Kuhn, eds., *Boundary Elements IX*, *Vol 1: Mathematical and Computational Aspects*, 125-138. Computational Mechanics Publications, Springer-Verlag, Southampton, 1987.

[7] S. Timoshenko, J.N.Goodier. *Theory of Elasticity.* McGraw-Hill, New York, 2nd ed., 1951.

[8] J.P. Den Hartog. *Advanced Strength of Materials*. McGraw-Hill, New York, 1952.

[9] A.C.A. Ramsay, J.P.B. Moitinho de Almeida, E.A.W.Maunder. Curious convergence with hypostatic hybrid equilibrium models. *Commun. Numer. Meth. Engng.*, **13**: 541-552, 1997.

[10] R.D. Cook, D.S. Malkus, M.E.Plesha, R.J. Witt. *Concepts and applications of finite element analysis.* Wiley, New York, 4th ed., 2002.