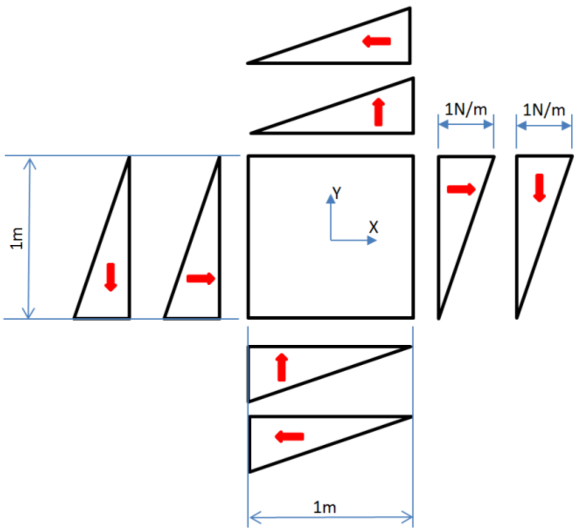
NAFEMS The NAFEMS Benchmark Challenge

Stress at the Centre of a Square Plate with Linear Boundary Tractions

The Challenge

A unit square homogeneous and isotropic steel plate is centred at the origin of the XY-plane with edges parallel with the coordinate axes and is loaded with linearly distributed normal and tangential boundary tractions as shown in figure 1. The plate can be assumed to be thin so that a plane-stress constitutive relationship is appropriate and for convenience a unit thickness may be used.



The Challenge

The challenge is to produce <u>two models</u> of this problem in your finite element software and then answer some questions. The first model should use a <u>single four-noded element</u> and the second a <u>single eight-noded element</u>. As engineers interested in the integrity of the plate we might wish to see the distribution of von Mises stress over the plate. We would like you to provide:

a) Numerical values for the von Mises stress at the centre of the plate for both models,
b) A statement as to which of these values is correct,
c) Contour plots of von Mises Stress for both models,
d) A brief commentary on how you modelled the problem and what, if anything, of interest you note about this problem – please include details of the software that you used.

Figure 1: Benchmark Challenge Number 1

Raison d'être for the Challenge

This challenge derives from a philosophical question; *can a problem be specified where the finite element response is null?* At first sight it seems a little improbable that such a problem can be conceived. However, when one realises that the boundary tractions are applied to the model in the form of consistent nodal forces and that if suitable tractions are chosen such that the consistent nodal forces cancel out then such a problem is easily found. This is the case for the challenge problem when a single four-noded element is used. Further consideration of the problem shows that it possesses a theoretically exact solution which involves linear stress fields that can be captured exactly with a single eight-noded element. The theoretically exact von Mises stress at the centre of the plate is zero and therefore both the single four-noded element and the single eight-noded element predict this value correctly. Disappointingly, however, it will be seen that even though the exact solution is recovered for the single eight-noded element, the available post-processing facilities in many commercial finite element systems will not allow the user to appreciate this fact because they use linear-interpolation of nodal stress values to simplify plotting procedures.

Consistent Nodal Forces

Whilst the plate model is loaded with (linear) boundary tractions, the finite element model can only take loads at nodes. As such the traction distributions need to be converted to (statically) equivalent nodal forces and this *should* be done in a manner *consistent* with the finite element formulation. For the normal tractions considered here the four-noded element this process is identical to 'lumping' the resultant force at the nodes in a manner such that both the traction distribution and the nodal forces produce the same resultant force and moment. However, for the eight-noded element, a similar lumping approach leads to a non-unique conversion process where there is an unknown load α which can take on any positive or negative value.

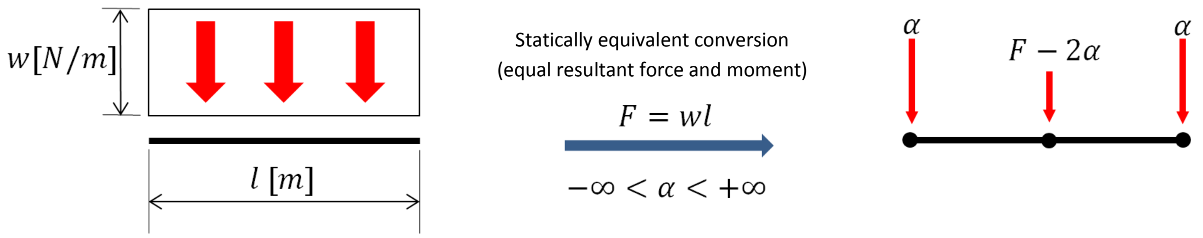
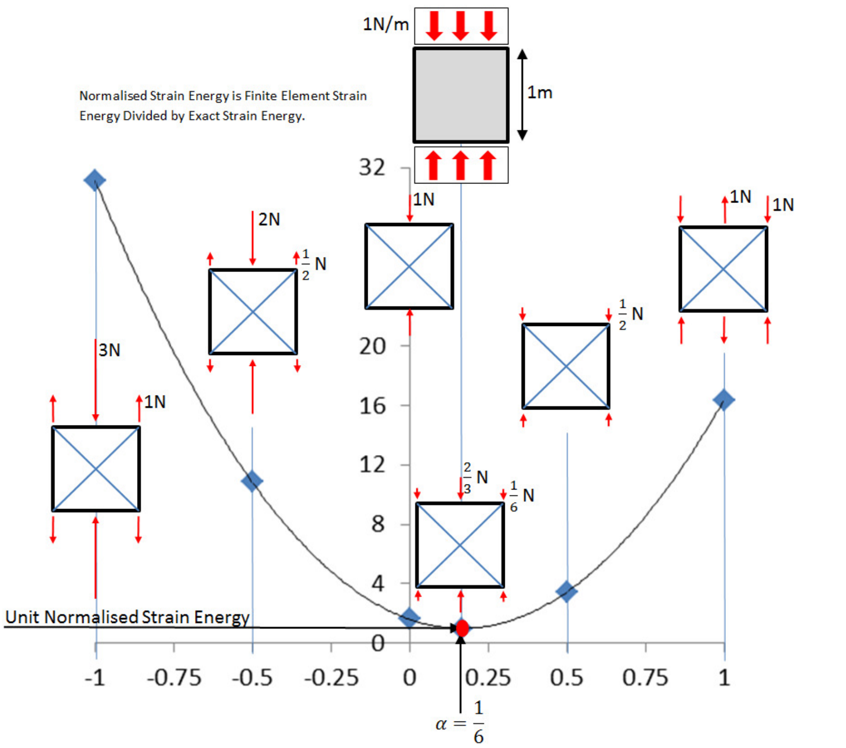


Figure 2: Statically equivalent conversion of a uniform traction to nodal forces on a quadratic edge

Different values of α will lead to different finite element solutions and this is demonstrated for a square plate with *uniform* normal tractions.



A unit square plate (with unit thickness) is loaded with uniform normal tractions of unit amplitude. This problem has a theoretically exact solution:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \\ 0 \end{Bmatrix}$$

The exact strain energy is defined in terms of Young's Modulus as:

$$U = \frac{1}{2E}$$

The figure show finite element results for the strain energy (normalised with the exact value) for six different values of α. The finite element model used four six-noded triangles and can capture precisely the exact solution.

Figure 3: Normalised finite element strain energy for different values of α

The exact strain energy (unit normalised strain energy) is seen to be captured when $\alpha = {}^{1}/_{6}$ which is also precisely at the minimum of the quadratic curve running through the six points. This is no accident as this value of α is that obtained by calculating it in a consistent manner with the quadratic shape functions. This example thus offers a convincing argument as to the reason why equivalent nodal forces should be generated in a consistent manner. If the reader does not believe this observation then he/she could easily run the models above and observe that the exact constant stress field is only captured when α takes on the value obtained using a consistent approach.

Let us look in a little more detail at one of the non-consistent or *inconsistent* sets of nodal forces, say the case where $\alpha = {}^{1}\!/_{3}$ as, in the absence of any knowledge of the correct consistent manner of calculating equivalent nodal forces this might be the most naturally assumed distribution to represent a uniform traction distribution. The nodal forces for this value of α can be decomposed into a sum of the correct consistent nodal forces plus an additional set of self-balancing nodal forces as shown in figure 4.

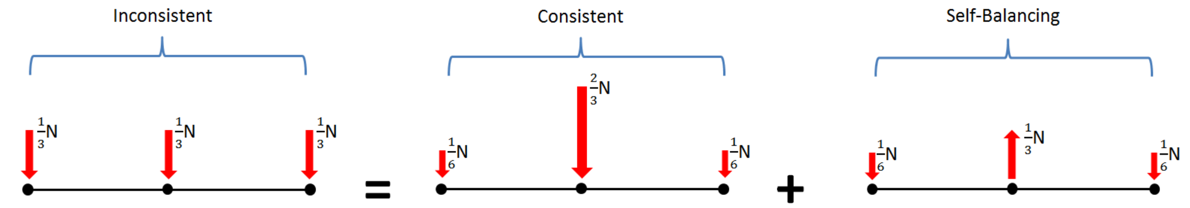
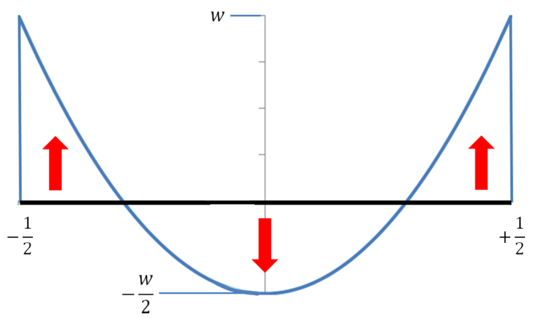


Figure 4: Decomposition of inconsistent nodal forces

Considering the self-balancing part of the inconsistent nodal forces one could ask the question; can a traction distribution be found for which these self-balancing nodal forces form a consistent set of nodal forces? The answer to this question is in the affirmative. The distribution is a quadratic (Legendre) polynomial as shown in figure 5.



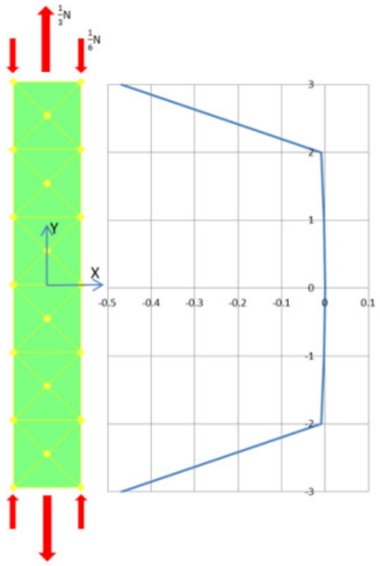
The quadratic traction distribution is a standard quadratic Legendre polynomial scaled by the factor *w*. The consistent nodal forces corresponding to this traction distribution are calculated in Appendix 1 so that the value of *w* appropriate for the self-balancing nodal forces in figure 4 can be determined as:

$$w = \frac{5}{2}$$

Figure 5: Quadratic self-balancing traction distribution

All the sets of inconsistent nodal forces shown in figure 3 may be decomposed in a similar manner to that described above with each set leading to different amplitudes for the self-balancing set of nodal forces and, therefore, different scale factor *w*. In other words, if one does not apply the boundary loading in a consistent manner then the model will be loaded with an additional set of self-balancing nodal forces consistent with the quadratic traction distribution of the form shown in figure 5.

What effect will this additional self-balancing load have on the model? Well in the first instance it is going to change the stress distribution local to the static boundary and, as shown in figure 3, this leads to an increase in the strain energy. However, as it is self-balancing then, by appeal to St Venant's Principal, one might hope that the influence of the self-balancing tractions on the stress distribution will decay with distance from the static boundary boundary. This effect is illustrated in figure 6 where the self-balancing nodal forces of figure 4 are applied to a longer version of the model shown in figure 3.



The model in this figure is the same as that shown in figure 3 except that the length in the Y direction has been increased to 6m to allow the stresses to diffuse. It is loaded with the self-balancing nodal forces shown in figure 4. The average value of the direct stress in the Y direction has been calculated at 1m intervals along the length of the plate and these have been plotted in the graph.

The effect of using an inconsistent set of nodal forces is clearly seen in that the average stress on the static boundaries is (erroneously) increased by nearly 50%. However, by St Venant, these erroneous stresses decay rapidly as one moves away from the static boundary.

Figure 6: Decay of stresses induced by self-balancing nodal forces

In a manner similar to the way in which traction distributions need to be applied as consistent nodal forces, for an appropriately restrained model the nodal reaction forces will emerge in the same form. So, for example, taking the 6m long plate shown in figure 6 and assuming that St Venant's principal does its job, then if the bottom edge were restrained from displacement in the Y direction and any of the sets of nodal forces shown in figure 3 applied to the top edge of the model, the reaction distribution at the bottom edge should correspond to the consistent set of nodal forces. It is therefore clear that some thought needs to be exercised when interpreting distributions of nodal reaction force since they can, potentially, be as confusing as the applied nodal force distributions.

Since calculating and applying consistent nodal forces is a tedious and potentially error-prone process, commercial finite element software tends to automate this for the engineer; the engineer applies boundary tractions and the software converts these, internally, into consistent nodal forces. To illustrate the values of these forces for the challenge problem they were calculated and are shown in figure 7.

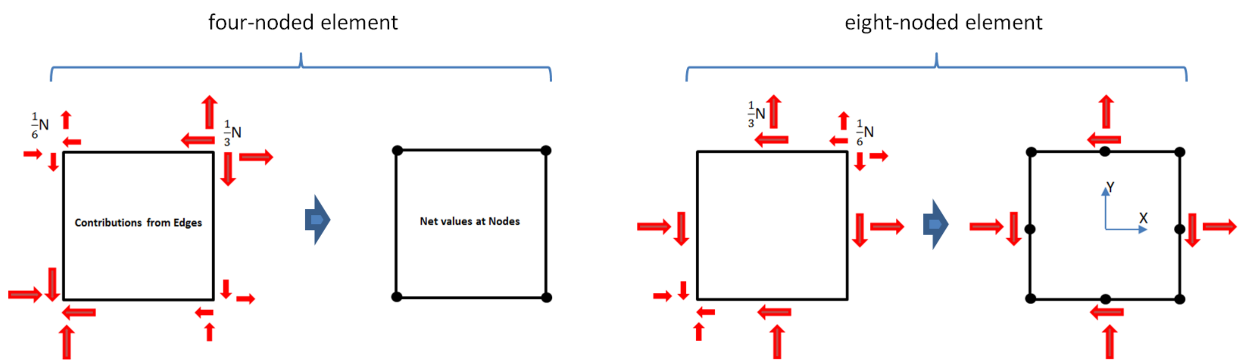


Figure 7: Consistent nodal forces for the two single element models

The reason why the single four-noded element produces a null result for this problem is now clear since the net nodal forces are all zero. This is not the case for the single eight-noded element which has non-zero nodal forces, at least for the mid-side nodes, and these thus lead to a non-null finite element solution which, as will now be shown, is the exact solution.

The Theoretically Exact Solution

The challenge problem was conceived through specifying a set of boundary tractions which held the plate in equilibrium and led to zero net consistent nodal forces for the single four-noded element model. The tractions are linear along the model boundaries and the theoretically exact solution must have stresses that equilibrate with these tractions. The finite element results for the single eight-noded element exhibit such linear distributions on the boundaries and these stress fields are shown in figure 8 together with the boundary tractions corresponding to each component of stress and expressions for the stresses that fit the contours and equilibrate the boundary tractions.

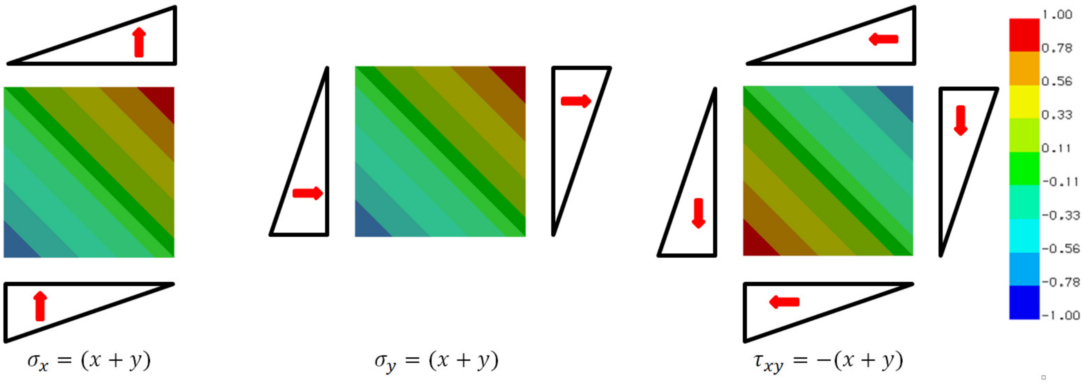


Figure 8: Stress field and corresponding boundary tractions (eight-noded element model)

Two further checks will show that the stress field shown in figure 8 is the theoretically exact one:

a) The challenge problem involved no body loading and so the exact stress field must reflect this. This can be confirmed by checking that the internal equilibrium conditions of the second column of table 1 are satisfied.
b) The strain field corresponding to the exact stress field will be compatible with a continuous displacement field. The strain compatibility conditions are given in the third column of the table and involve second derivatives of the strains. It is easily seen that these are satisfied since the stresses, and hence the strains, are of first degree (linear) so that all second derivatives are zero.

Postulated Stress Field	**Equilibrium Conditions**	**Compatibility Conditions**
$\sigma_x = (x+y)$ $\sigma_y = (x+y)$ $\tau_{xy} = -(x+y)$	$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$ $\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0$	$\frac{\partial_2 \varepsilon_x}{\partial y^2} + \frac{\partial_2 \varepsilon_y}{\partial x^2} - \frac{\partial_2 \varepsilon_{xy}}{\partial x \partial y} = 0$

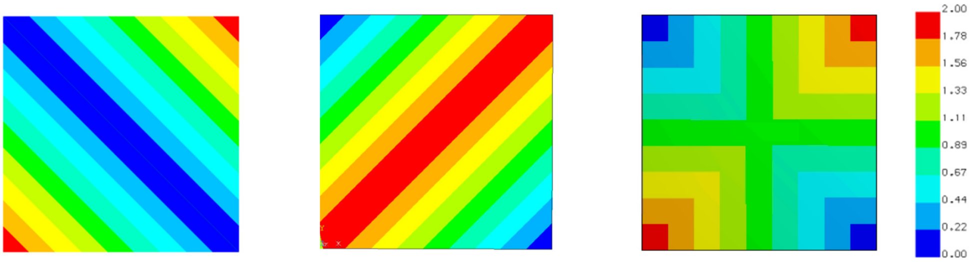
Table 1: Postulated stress field and (Internal) conditions for a theoretically exact solution

Low Fidelity of Post-Processing in Commercial FE Software

Whilst the single eight-noded element was able to capture the exact solution to the challenge problem, the low-fidelity of the post-processing in many commercial finite element systems means that this fact is not immediately evident. For example whilst the linear stress components are plotted correctly, the von Mises stress, which is piecewise linear about the line $x + y = 0$, is not.

$$\sigma_{vM} = 2\sqrt{(x+y)^2} = 2|x+y|$$

The contour plots shown in figure 9 give the exact distribution and that offered by two versions of the same popular commercial software. The same contour range and colours are used for all three plots.



Exact CS 1 CS 2

Figure 9: Contour plots of von Mises stress for the challenge problem

If the von Mises stress at the centre of the plate is read from the contour plots then the values for the two versions of the commercial software would be 2 and 1 compared with the exact value of zero!

Discussion

The reason that a problem *could* be conceived where the finite element response is null is a characteristic of the standard displacement formulation used in the majority of commercial finite element systems. The formulation requires traction distributions to be converted into consistent nodal forces and this process is one where certain information about the original boundary tractions is lost.

This idea can be illustrated by attempting to apply the quadratic traction distribution of figure 5 to the edge of a four-noded element. Lumping according to equilibrium requirements (equal net force and moment) leads to the consistent nodal forces for this element and so it is easily seen that the consistent nodal forces for this traction distribution are zero – see Appendix 1. Thus any amount of this quadratic traction distribution could be added to the edge of an (four-noded) element without the element feeling it. This is simply a manifestation of what is called *weak equilibrium* – whilst the correct resultant forces and moments are transmitted through the elements by nodal forces, point-by-point equilibrium is not generally satisfied for coarse meshes.

A further manifestation of the loss of information that may occur through the conversion of traction distributions to consistent nodal forces is that concerning the particular edge on which the traction was applied. This was the case with the challenge problem where for both element types the nodal forces at the corners of the plate cancelled out.

For the challenge problem, then, we see that whilst the single eight-noded element recovers the exact solution, the single four-noded element is incapable of responding despite applying the loads in the correct manner. The resulting null stress field does not equilibrate with the applied boundary tractions. As the finite element method is an approximate method some form of weakening of the solution has to be accepted. For the standard displacement element this weakening occurs in the equilibrium of the resulting stress fields as has been shown.

Whilst a coarse mesh of displacement elements might show significant violations of equilibrium, mesh refinement generally leads to finite element results that are close to equilibrium in a point-by-point sense. There are two distinct types of mesh refinement that can be performed and these are known as *p*-refinement and *h*-refinement. With *p*-refinement the number of elements and geometric arrangement of the mesh is kept constant whilst the degree of approximation in each element is increased. The process of changing from four-noded elements to eight-noded element, as done for the challenge problem, is an example of *p*-refinement and in this case convergence was rapid – from a null solution to the exact solution. For the challenge problem, *h*-refinement can be demonstrated by performing uniform element subdivision of the four-noded element. The results for this process are summarised in figure 10 which shows the convergence of strain energy and the maximum von Mises stress with number of degrees of freedom. One might be surprised, and possibly worried, that even for the most refined mesh an error of nearly 3% remains in the maximum von Mises stress!

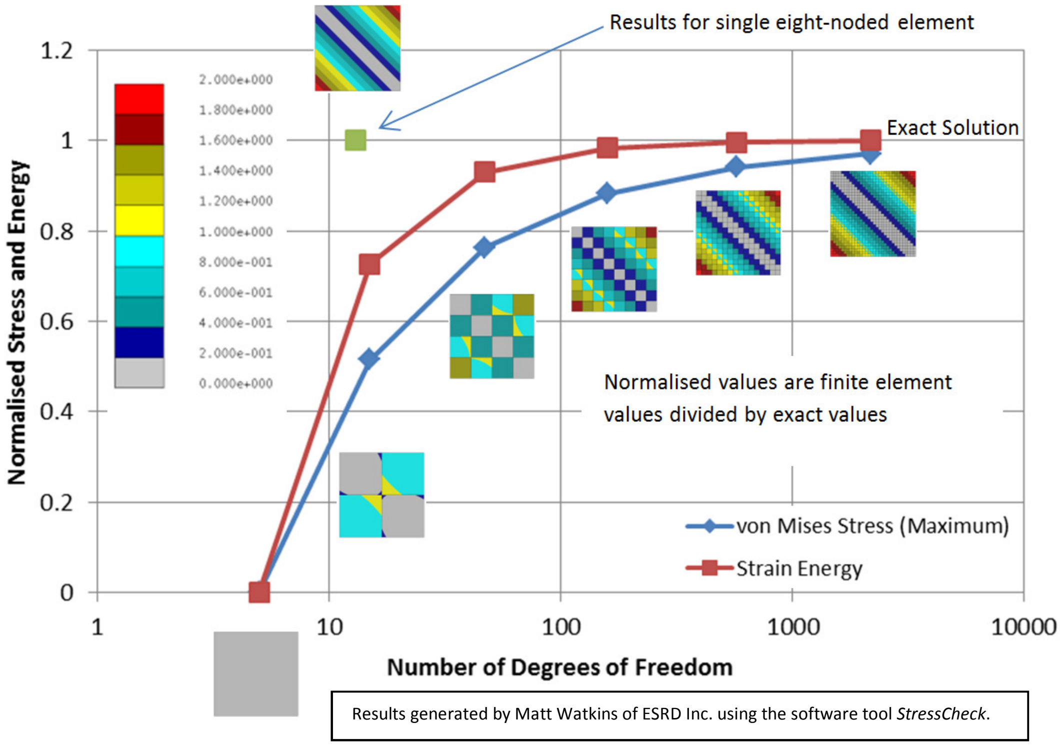
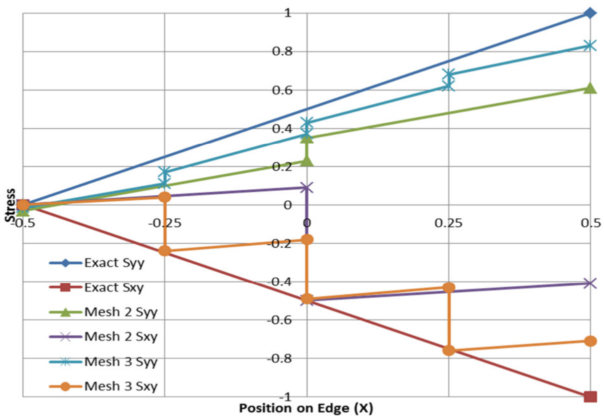


Figure 10: Convergence (*h*-type) for the four-noded element

The way in which weak equilibrium is gradually pushed towards strong point-by-point equilibrium can be demonstrated further by plotting pertinent components of the finite element stress field and comparing these with the prescribed tractions. This has been done in figure 11 for the top edge of the challenge problem and for the first two levels of *h*-refinement after the single four-noded element. The finite element stresses were unaveraged and show significant discontinuities at the nodes.



This figure plots the two components of stress, Syy and Sxy, along the top edge of the model for meshes 2 and 3. For equilibrium these stresses should equilibrate with the stresses corresponding to the applied tractions (shown in the figure as 'exact'. Whilst the finite element stresses are converging towards the exact distribution for these coarse meshes there is a significant violation of equilibrium both in a point-by-point sense and also in an integral sense – the resultant force of the finite element stress distributions is significantly different from the applied resultant force.

Figure 11: Convergence of finite element boundary stresses for four-noded models

Closure

This challenge problem, which started life as a philosophical quest to find a real problem for which a valid finite element method would respond with a null solution, has led, via a canter through consistent nodal forces and exact solutions, to a conclusion which most seasoned FE analysts will be familiar with. This conclusion is that finite element solutions are approximations of the truth and that with conventional conforming elements it is strong equilibrium that is compromised. The solution to this issue is to ensure that the mesh is suitably refined so that the quantity of interest can be demonstrated to have converged to within sufficient accuracy to the exact solution. The level of refinement required for reasonable engineering accuracy might be quite significant as seen in figure 10 for the relatively simple challenge problem.

The challenge problem has highlighted a new form of finite element error that the user needs to be aware of - this being the inability of some commercial software to render the results correctly. This form of error might be termed the 'post-processing fidelity error'. This error caused a number of the challenge responders to report an incorrect value for the von Mises stress at the centre of the plate. The reason for this error is that most commercial finite element systems opt to use some form of linear interpolation of nodal results to define the field within the element for the purposes of contour plotting. This is not appropriate for the displacement fields of elements with quadratic interpolation functions and also for the stress components which may contain quadratic terms. It is certainly inappropriate for principal quantities and stress invariants which, by definition, may be non-linear and might even, as in the challenge problem, contain gradient discontinuities.

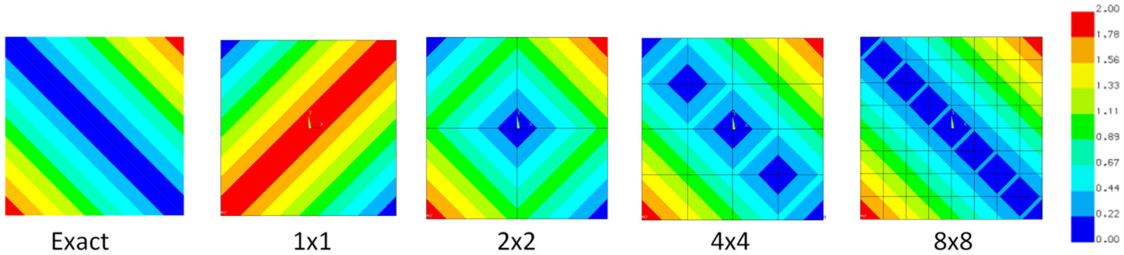


Figure 12: Post-processing fidelity error in von Mises stress

The contour plots shown in figure 12 are the von Mises stress produced by the commercial software CS1 for increasingly refined meshes of eight-noded elements – note that the actual finite element solutions are exact and identical for each of these meshes.

Of the respondents to the challenge three provided answers that would withstand the scrutiny of a design review or audit. The first of these used mesh refinement to demonstrate that the result for the eight-noded model could be reasonably assumed to be the exact solution whilst the second recognised immediately that the problem had a theoretically exact solution independent of any finite element approximation. The third respondent recognised that a change of coordinate system could transform the challenge problem into a more familiar one, namely, that of pure bending – see figure 13. Having mentally performed the transformation the respondent then used his understanding of how the standard four-noded and eight-noded elements perform under pure bending to *state* the results (correctly) without recourse to finite element software. This response can only have come from someone with an intimate knowledge of continuum mechanics and the finite element method!

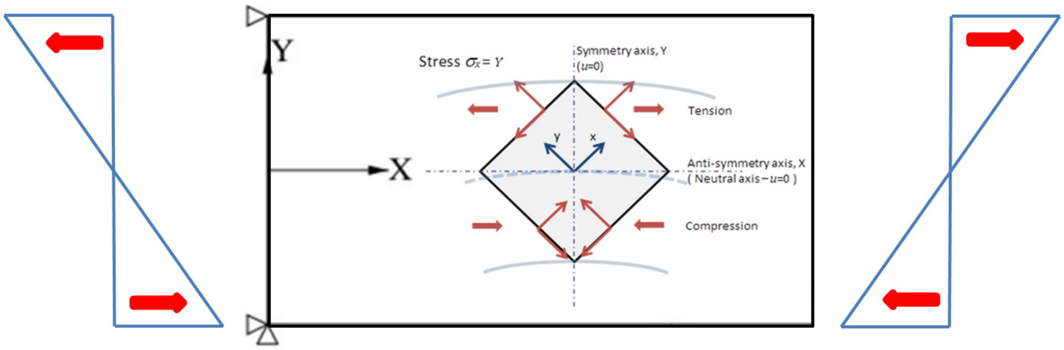


Figure 13: One respondent's transformation of the challenge problem

Other respondents guessed at the correct result and some guessed correctly despite the low-fidelity of the finite element results they were presented with (see figure 9). However, a reviewer adopting the philosophy of the Napoleonic Code – '*Guilty until Proven Innocent*', is unlikely to be satisfied with such a response.

Producing finite element results that have not been verified is just one of many forms of *finite element malpractice* that occur in the industry. Many times it goes unquestioned and components and structures are built based on erroneous analysis results. Often, where sufficient ductility exists, the inherent redundancy of a structure will allow the stresses and therefore the internal forces to redistribute safely. Other times, however, this can lead to a major disaster as in the case of the Sleipner Drilling Platform where an extremely crude and unrefined finite element model predicted stresses some 45% below the actual value. So it is incumbent on the engineer and his/her manager, if they want to rest well at night, to ensure suitable verification has taken place.

Appendix 1: Consistent Nodal Forces for a Self-Balancing Traction

Consistent nodal forces for a self-balancing quadratic traction distribution are calculated for edges of a four-noded element and an eight-noded element. The traction distribution, t, is shown for a unit edge in figure 14. The distribution is the basic quadratic Legendre polynomial scaled by a factor w. Two edge ordinates are shown with s having length dimensions and ξ being non-dimensional. The shape functions are given with the superscript indicating the number of nodes for the element.

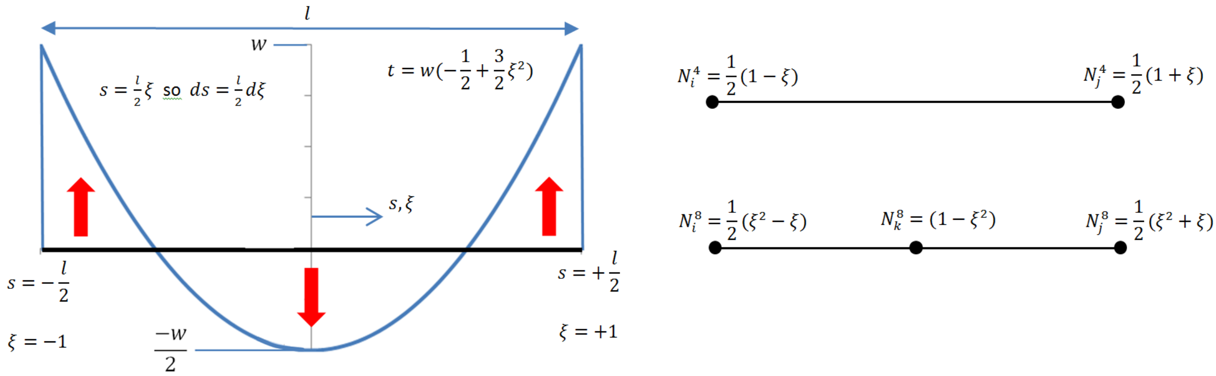


Figure 14: Self-balancing traction distribution and four-noded and eight-noded element edges

The expression for a consistent nodal force is:

$$q_n^p = \int_s N_n^p t\, ds = \frac{l}{2}\int_{-1}^{+1} N_n^p t\, d\xi$$

The consistent nodal force at node i of an edge of a four-noded element is zero and, by symmetry, the consistent force at node j is also zero:

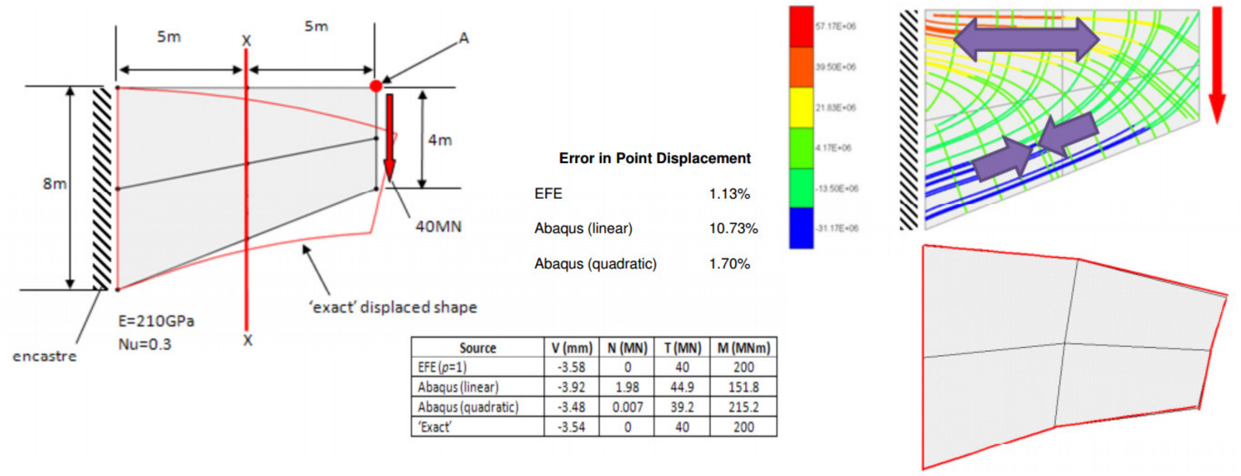
$$q_i^4 = \frac{l}{2}\int_{-1}^{+1} \frac{1}{2}(1-\xi)w\left(-\frac{1}{2}+\frac{3}{2}\xi^2\right)d\xi \;=\; \frac{lw}{8}\int_{-1}^{+1}(-3\xi^3+3\xi^2+\xi-1)d\xi \;=\; \frac{lw}{8}[-\frac{3}{4}\xi^4+\xi^3+\frac{\xi^2}{2}-\xi]_{-1}^{+1} = 0\,;\; q_j^4 = 0$$

The consistent nodal forces at all nodes on the edge of an eight-noded element are non-zero. The consistent nodal force is calculated for the central node and the corner node forces are deduced by equilibrium considerations:

$$q_k^8 = \frac{l}{2}\int_{-1}^{+1}(1-\xi^2)w\left(-\frac{1}{2}+\frac{3}{2}\xi^2\right)d\xi \;=\; \frac{lw}{4}\int_{-1}^{+1}(-3\xi^4+4\xi^2-1)d\xi = \frac{lw}{4}[-\frac{3}{5}\xi^5+\frac{4}{3}\xi^3-\xi]_{-1}^{+1} = \frac{-2lw}{15}\,;\; q_i^8 = q_j^8 = \frac{lw}{15}$$

Appendix 2: Results from an Equilibrium Plate Membrane Element

Following the spirit that these challenge problems should be educational, some results generated using a *p*-type equilibrium element are presented along with the corresponding standard displacement element results. The problem is a deep tapered cantilever that is built-in at the deep-end and loaded with a uniform tangential traction at the free-end. A mesh of four quadrilateral elements is used and the results are presented in terms of the vertical displacement, V, at point A and the stress resultants along X-X. For the equilibrium element, linear statically admissible stress fields (*p*=1) were used whilst for the displacement element both four-noded and eight-noded elements were used. This problem has no known closed-form exact solution so that the results presented as 'exact' are obtained from a highly refined and converged finite element model.



Source	V (mm)	N (MN)	T (MN)	M (MNm)
EFE (*p*=1)	-3.58	0	40	200
Abaqus (linear)	-3.92	1.98	44.9	151.8
Abaqus (quadratic)	-3.48	0.007	39.2	215.2
'Exact'	-3.54	0	40	200

Figure 15: Deep tapered cantilever problem with results

The displacements from a displacement element are continuous at the nodes whereas for an equilibrium element they are not. The displaced shape for the equilibrium element has the displaced edges shown in red and the nodal averaged displaced shape shown in grey. A small discontinuity can be seen but nonetheless the average displacement is at least as accurate as that produced by the displacement elements.

The stress resultants N (normal), T (tangential) and M (moment) on X-X are calculated by integrating the finite element stresses along the section. We have seen already that displacement elements do not generally, particularly for coarse meshes, satisfy equilibrium and the discrepancy in the stress resultants between the displacement element results and the exact values is not insignificant! For the equilibrium element, even though the finite element stresses are not the exact values, provides an equilibrium set of stress resultants. Since the equilibrium elements produce statically admissible stress fields other quantities such as principal stress trajectories may also be simply plotted. This sort of plot gives a complete description of the way in which the loads are transmitted through the structure in the form of stresses. Stress resultant arrows have been added to the figure to indicate the main compressive and tensile load paths and these might be further used to think of the model as a strut and tie system.