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The Case for Equilibrium Models in FEM with Application to Slab Design

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Summary

A case is presented for the use of equilibrium models as an alternative, or a complement, to conventional conforming models in finite element analysis. A short summary is given of the formulation of equilibrium models based on hybrid stress elements of variable degree, and then they are considered in the context of modelling concrete slabs governed by Reissner-Mindlin plate theory for linear elastic behaviour. Examples are presented of a flat slab and the problems of modelling its column zones, and a simply supported skew bridge deck. The latter is also considered as a yield line problem, and a safe solution procedure is proposed based on the use of triangular finite elements which can provide both upper and lower bounds to the collapse loads.

Keywords: Equilibrium models; finite elements; concrete slabs.

1. Introduction

Conventional finite element models based on conforming displacement fields have been used for some 50 years, and they have reached a fair state of maturity. Their development took precedence over stress based elements mainly due to their relative simplicity in formulation. However it is considered that, from the structural engineers point of view, greater insight to help engineering judgement would be better given by knowledge of equilibrium of stress rather than conformity of deformations.

Strong forms of equilibrium are desirable and achievable, and suitable models were initially developed by Fraeijs de Veubeke et al in the Liège School [1,2], however due to their relative complexity, they have made little impact on the industrial user. Recent work by the authors has focused on developing a finite element system based on *p*-type hybrid stress equilibrium elements, and this paper explains how and when they might be useful for the design of concrete slabs.

Section 2 summarises the formulation of such elements, and Sections 3 and 4 present examples of their application to flat slabs and skew bridge decks respectively. Further discussion and concluding remarks are made in Section 5.

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EFE Equilibrium Finite Elements
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2. Formulation of equilibrium elements

Equilibrium elements are here described as a type of hybrid stress element of variable polynomial degree for linear elastic plate problems [3]. Internal stress fields σ and σ_p are defined to be statically admissible with zero body forces, and with body forces as a particular solution, respectively. t and t_p denote the corresponding equilibrating side tractions, and $\bar{t}$ denotes prescribed tractions or interactions.

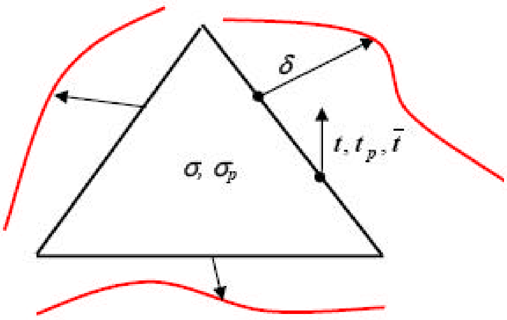

Fig. 1 *Primitive hybrid element.*

The term "stress" is used here as a generic term for plate stress-resultants involving bending and torsional moments of degree p, and shear forces of degree $(p-1)$.

Side displacement fields δ are defined independently for each side, and hence are discontinuous at the corners of the element as indicated in Figure 1.

These fields are defined in terms of rotations and deflections in accordance with Reissner-Mindlin theory [4], with complete polynomials of the same degrees as for the moment and shear fields respectively.

Stress, traction, and displacement fields for a single element are expressed in terms of parameters s and v according to Equation (1),

$$\sigma = Ss;\ t = \bar{S}s;\ \delta = Vv \tag{1}$$

and they are related by conditions of weak compatibility and potentially strong equilibrium by Equations (2) and (3) respectively [5], where f denotes the matrix of the constitutive relations.

$$\int_{\Omega} S^T \left(f\sigma + f\sigma_p\right) d\Omega = \oint_{\partial\Omega} \bar{S}^T \delta \cdot d\Gamma \ \text{ or } \ F \cdot s - D^T v = \int_{\Omega} S^T f\sigma_p d\Omega \tag{2}$$

$$\oint_{\partial\Omega} V^T \left(t + t_p\right) d\Gamma = \oint_{\partial\Omega} V^T \bar{t} \cdot d\Gamma \ \text{ or } \ Ds = \oint_{\partial\Omega} V^T \left(\bar{t} - t_p\right) d\Gamma \tag{3}$$

Strong equilibrium occurs when body loads and prescribed tractions are equilibrated by stress fields of the same degree as σ. Thus plate elements of degree 2 will satisfy strong equilibrium with uniformly distributed loads. Elimination of stress parameters s from Equation (2) leads to a stiffness matrix $K = DF^{-1}D^T$ for a primitive element which is generally rank deficient due to the existence of both rigid body modes and spurious kinematic modes. However stable elements, free of spurious kinematic modes, can be defined by assembling primitive elements, e.g. into triangular or quadrilateral forms termed macro-elements [1,6,7].

Although both statically admissible stress fields and displacement fields are defined in a discontinuous way for the macro-elements, the solutions from complete finite element models can exhibit good levels of continuity as indicated in later examples. It should be noted that a conventional displacement element, such as the 6-noded triangle based on rotation and deflection fields both of degree 2, has derived fields of moment and shear of degree 1 and 2 respectively, which violate equilibrium conditions with uniformly distributed loads and equilibrium between moments and shear forces! Thus compatibility is achieved at the expense of equilibrium.

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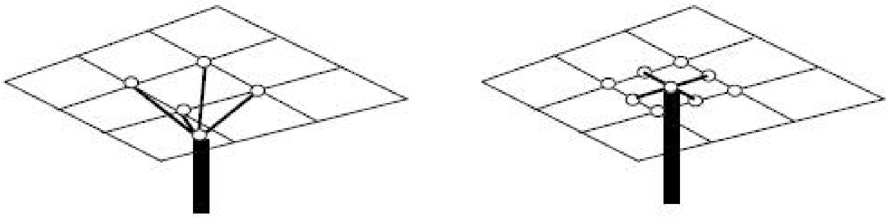
3. Flat slabs

The modelling of flat slabs in reinforced or prestressed concrete construction poses challenging problems, particularly for the simulation of column zones. Modelling with 3D solid elements is generally avoided as being unnecessarily complex and demanding on computational effort. Various combinations of beam and plate elements have been used with displacement models [8,9]. These range from direct connection of 1D column elements with nodes of plate elements to use of multi-point constraints. Some examples are illustrated in Figure 2.



(a) (b)

Fig. 2 *Typical models for column zones using displacement elements.*

In (a) the column head is modelled with 4 bar or beam elements connected to 4 corner nodes of a plate element. In (b) the connections are made in the plane of the plate with 4 link elements that connect to the midside nodes of a plate element. The links may be flexible or rigid. Alternatively, nodes situated within the cross-section of a column may be constrained to conform with the rigid body displacement of a column head. Such models can satisfactorily represent the global behaviour of the slab, but they cannot give a realistic picture of the local actions around a column head. Multi-point constraints can also lead to singularities, e.g. at the corners of column sections, and these can pollute the quality of a solution for moments and shear forces at critical sections of design.

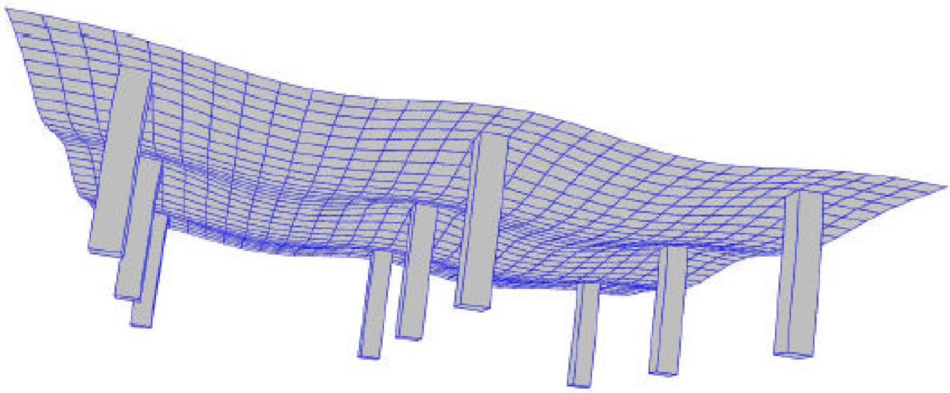


Fig. 3 *Perspective view of a deformed flat slab and supporting columns.*

An equilibrium model is proposed which involves similar connections as in (b), but now corner nodes are only used to specify shape and position, and midside nodes serve as reference points for connection and interaction via side modes of traction or displacement. The stiffness of the column head depends on the properties of the link elements, the plate element which acts in parallel, and their interactions. The distributions of stress-resultants in the plate elements which surround the perimeter of the column section balance the stress-resultants in the column whether theoretical

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singularities exist or not, and thus provide the designer with a much more complete picture of this critical zone.

A qualitative example is illustrated in Figures 3 and 4. These concern a slab supported on a regular grid of columns with square cross-sections. The plate is modelled by a uniform mesh of square macro-elements of moment degree 2, with the element size chosen to match the column section. Element dimensions could be increased in zones between the columns using a non-uniform mesh. The columns are fixed at their bases, and the slab is subjected to a uniformly distributed load.

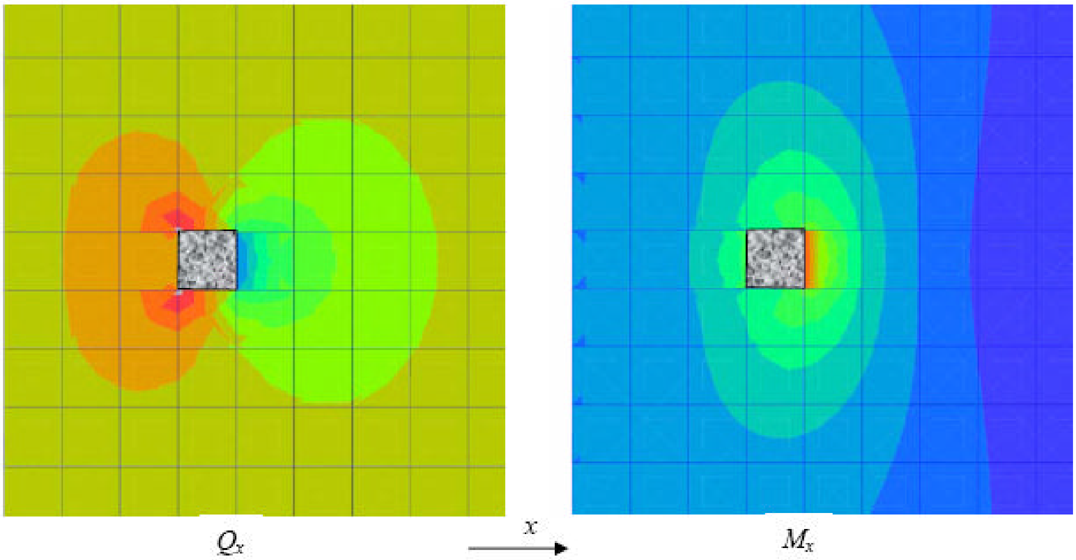

Fig. 4 *Contours of shear force Qx and moment Mx in an edge column zone.*

The zone illustrated in Figure 4 is sectioned from the slab with its free edge on the left hand side.

4. Skew slabs

This section serves to illustrate the potential for elastic and limit analyses in the context of a skew bridge deck. The deck is considered as an 0.75m thick concrete slab with a simply supported single span, a 45° skew, and a uniformly distributed load of 10kN/m^2 over the area CBEF in Figure 5(a), which represents an outer lane of traffic. A single span is likely to be supported by discrete bearings, e.g. stiff roller bearings or more flexible elastomeric bearings, without further means to provide torsional moments along the supported edge. Thus, in the context of Reissner-Mindlin plate theory, soft simple support conditions [4] are appropriate for simulating real supports.

4.1 Linear elastic analysis

Concrete material properties are assumed to be: $E = 28\text{kN/mm}^2$ and $\nu = 0.2$, and both equilibrium and conforming finite element models are considered. The equilibrium model consists of the mesh of triangular macro-elements shown in Figure 7(a) based on moment degree 4 and shear degree 3. Figure 5(b) shows a contour plot of torsional moments M_{xy} in the neighbourhood of corner C, where

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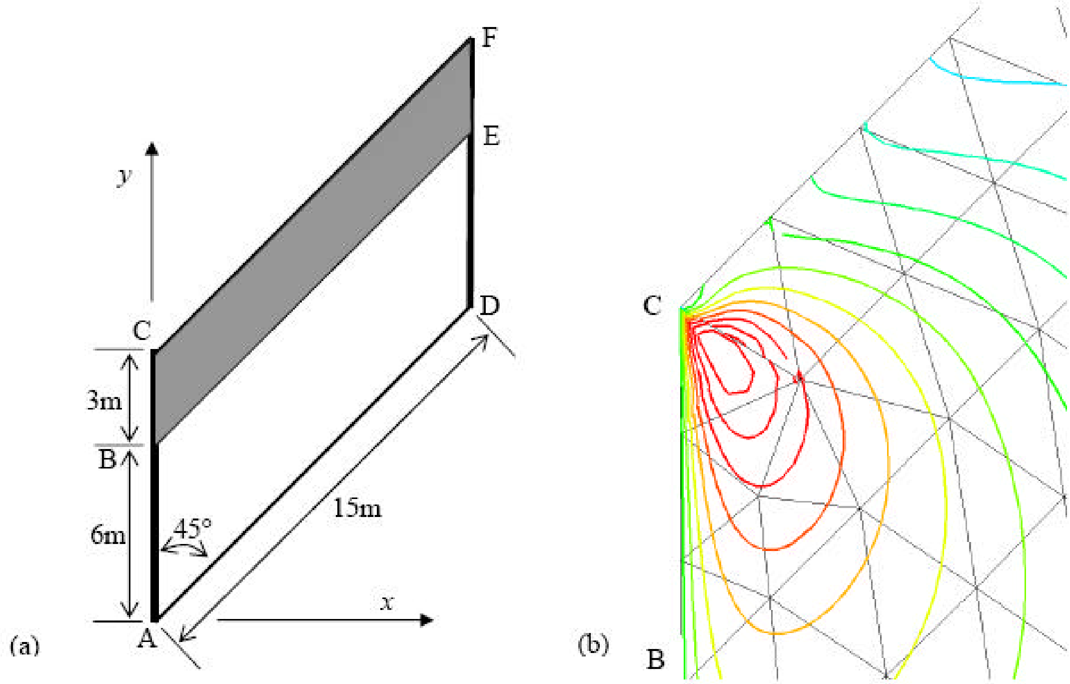

Fig. 5 *Skew slab and contours of torsional moments Mxy.*

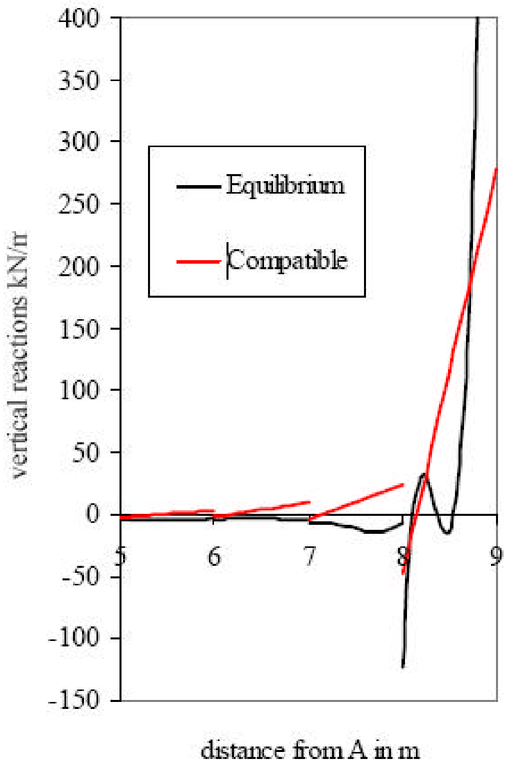

Fig. 6 *Vertical reactions along part of AC.*

a concentration of moment and large moment gradients are observed within a boundary layer. The maximum value of the moment is given as 37.64kNm/m. This solution is not of course exact, but it has two nice features: it satisfies local as well as global equilibrium, and the contours are mainly smooth and continuous.

From the theoretical point of view, singularities exist at the obtuse corners for shear forces, i.e. these forces become unbounded [10]. The singularities are consequences of the plate theory for a rigid line of support. More realistic results would need to allow for flexible supports, non-linear material behaviour, and the possibility of uplift as a non-linear contact problem. However it is useful to compare the finite element reactions with those from a compatible finite element model. These are shown in Figure 6 for the 4m adjacent to corner C. The compatible model is based on a uniform mesh of quadratic triangular elements with similar dimensions to those of the equilibrium model.

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The two distributions of reactions are shown as they are derived for element sides of length 1m. The equilibrium tractions are piecewise cubic, and the compatible tractions are essentially piecewise linear as derived from element stress fields. It is clear that both distributions are significantly affected by the singularity, and tend to oscillate with discontinuities. Similar discontinuities might be expected if the bearings themselves are discrete.

The equilibrium model has the advantage in this situation in that it provides a statically admissible solution to the designer, whereas the compatible model is more severely polluted by the singularity. Although the nodal forces equilibrate with the loads, the derived tractions have a resultant magnitude which falls short of that required for overall equilibrium by some 27%. This feature of compatible models is not uncommon, and can lead to disastrous consequences as experienced by the Sleipner offshore platform [9].

Although compatibility of deformations is only enforced in a weak sense in the equilibrium model, nevertheless the results for deflections can be of good quality, e.g. the maximum deflections along the free edge CF are output from the equilibrium and compatible models equal to 1.586mm and 1.573mm respectively. A similar quality is evident in Figure 3 for the deflections of the flat slab.

4.2 Limit analysis

An equilibrium model provides the possibility of implementing a form of Hillerborg's strip method [11] based on finite elements rather than strips, and including torsional moments. Moment redistribution can be effected using hyperstatic variables associated with patches of elements forming stars, i.e. elements that share a common vertex. A mathematical programme is then required to optimise a statical solution.

The yield line method can also be implemented using triangular finite elements [12,13]. In this case the elements act as rigid regions bounded by potential yield lines. The work method can be used whereby basic mechanisms defined by star patches of elements are combined so as to minimise the load factor of a given load distribution subject to yield constraints on the bending moments along the sides of the elements. This can be formulated as a linear programme, which satisfies the Kuhn-Tucker conditions.

An example is shown in Figure 7(a) for the skew bridge deck problem after meshing with the same Delaunay triangulation scheme that was used for the elastic analysis. The yield line pattern and the overall results are mesh dependent, but the final pattern for this mesh with an isotropic slab is as shown. This requires a yield moment of 46.32kNm/m to sustain the applied load, and the solution is compared with a simpler hand solution in Figure 7(b) where the yield moment increases to 47.45kNm/m. An alternative hand solution based on a single sagging yield line at midspan requires a yield moment of 46.85kNm/m.

Although the finite element yield line method has provided the most unsafe of the three solutions in this case, it should be emphasised that its solution also contains sufficient information to recover a complete statically admissible stress field. This follows since the solution of the primal linear programme leads to the solution of the dual programme in terms of element side moments and corner forces, which are in equilibrium. From this state of weak equilibrium, the nodal forces can be resolved into pairs of side forces, which are then replaced by statically equivalent distributions of traction in terms of shear forces and/or torsional moments [14,15]. A strongly equilibrated solution can then be determined by analysing each element separately subject to static boundary conditions. The resulting stress fields can now be surveyed and checked for any violations of the yield criterion so that safe conclusions can be made. Such stress fields are of course not available from conventional yield line analyses.

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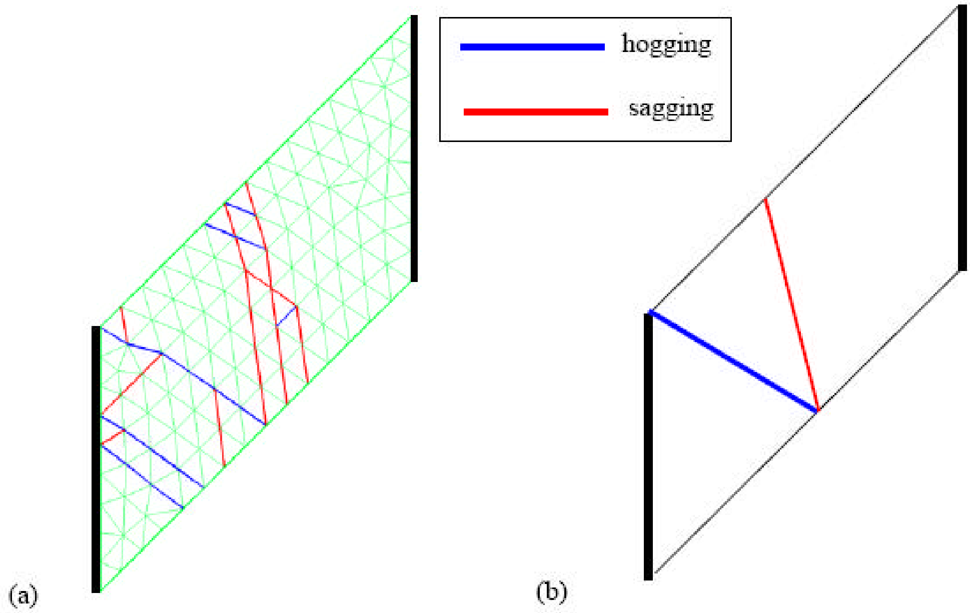

Fig. 7 *Yield line patterns from (a) an FE model, and (b) a hand analysis.*

5. Conclusions

Equilibrium finite element models have undergone considerable development since the pioneering work of the Liège School in the 1960s and 70s. Dual solutions obtained from compatible and equilibrium models are now becoming feasible, and this paper has demonstrated their use in the context of the design or assessment of concrete slabs.

With growing concerns for verification and validation issues, the need for performing at least two analyses, preferably from different types of model, is considered essential. In any event, the equilibrium finite element models provide first and foremost comprehensive information on statically admissible stress fields, which should be of primary interest to structural engineers.

Stable equilibrium elements for plates have been formulated in triangular and quadrilateral forms. Their incorporation into an existing finite element system based on a conventional stiffness method has proved to be reasonably straightforward. Their development within a tailor-made system where full advantage of their formulation as variable degree *p*-type elements is now in progress, and a preliminary version of the system was used for Figures 5(b), 6, and 7(a).

Future work is intended to focus on the implementation of limit analyses using the equilibrium concepts described in this paper.

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