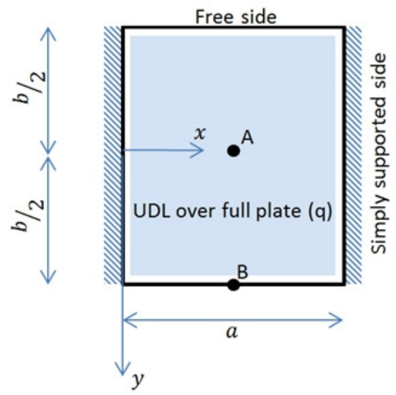
Poisson's Ratio for a Steel Plate

As a final year project, a student was asked to conduct an experiment on a thin steel plate, simply supported on two opposite sides and loaded with a uniformly distributed transverse load (UDL). The student was asked to measure the surface strains at the centre of the plate and from these to determine the value of Poisson's ratio for the steel plate.

The student recalled from an earlier lecture that if a plate-membrane is subjected to a uniaxial stress field then the ratio of the direct strain in the lateral direction to that in the longitudinal direction gave Poisson's ratio. Whilst she realised that because of symmetry the point at the centre would have zero shear stress, she was also aware that the state of stress at this point might well not be uniaxial. She recognised that for a force driven problem, such as this one, the stresses would be independent of the Young's modulus but that Poisson's ratio might well change the relative proportion of the two stress components. She also realised that if she took the ratio of the two stresses then this would be independent of the load magnitude. So, if she could find the true ratio of stress (or moments as these are proportional to the stresses) at the centre of the plate then her measured value should agree if her plate had the same value of Poisson's ratio as that used to evaluate the true ratio. For the true ratio she resorted to Timoshenko's text [1] which gives the moments at the centre of the plate for a value of Poisson's ratio of 0.3.



b/a	Point A: $x = a/2$; $y = 0$			Point B: $x = a/2$; $y = \pm b/2$	
	$w = \alpha_1 \frac{qa^4}{D}$	$M_x = \beta_1 qa^2$	$M_y = \beta_1' qa^2$	$w = \alpha_2 \frac{qa^4}{D}$	$M_x = \beta_2 qa^2$
	α_1	β_1	β_1'	α_2	β_2
0.5	0.01377	0.1235	0.0102	0.01443	0.1259
1.0	0.01309	0.1225	0.0271	0.01509	0.1318
2.0	0.01289	0.1235	0.0364	0.01521	0.1329
∞	0.01302	0.1250	0.0375	0.01522	0.1330

Figure 1: Challenge plate and table 47 from page 219 of Timoshenko's text [1]

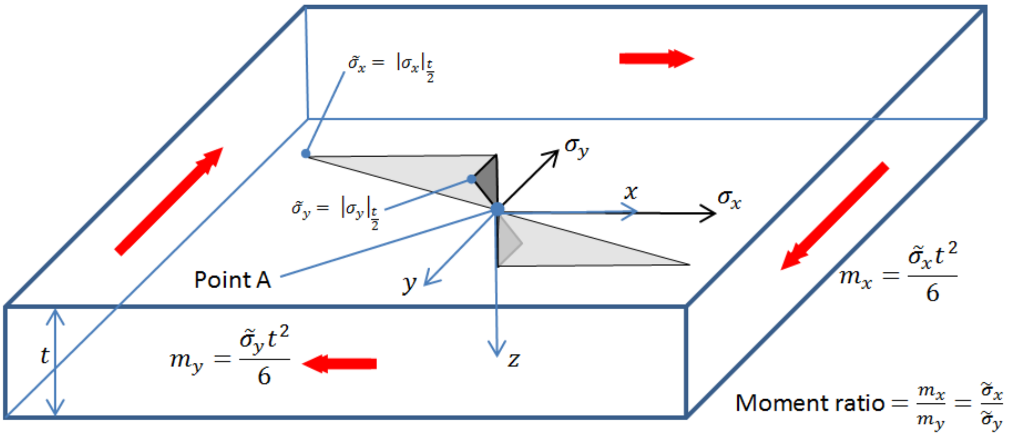
For her experiment she used a plate with an aspect ratio of b/a=0.5 and when she compared the ratio of the stresses she had evaluated (about 10) to that offered in Timoshenko's text (about 12) she realised that the steel of her plate must have a different Poisson's ratio to that of Timoshenko's value (0.3). At this point she was at a loss on how to continue and so she visited her supervisor for advice.

The Challenge

As this student's supervisor you were also surprised that the moment or stress ratios were so different particularly as you knew the value of Poisson's ratio for the plate to be very close to 0.3. You decided to try and understand the problem by modelling the plate in finite elements and investigating how the moment ratio changed with Poisson's ratio. The challenge is to conduct this numerical experiment and establish the truth!

Supervisor's Narrative

As this student's supervisor I am faced with unpicking the question she has posed me. I am pretty convinced that the plate should have a Poisson's ratio of 0.3 since this is the value specified by the supplier. So what can have happened? I think the first thing to do might be to undertake an exploration of the real solution for this problem and I can do this using a finite element model. We have commercial finite element software in the department so let's see if I can model this problem and I might also take a look at the NAFEMS booklet on how to use plate elements [2]. Since Timoshenko's solutions deals with moments and my student is looking at surface stresses then I prepared figure 2 to explain how the two are related and to point out that the stress ratio that my student is working with is identical to the moment ratio at the same point.



This figure shows an infinitesimal region around the centre of the challenge plate together with the moments and stresses. The uniformly distributed load causes sagging moments in both longitudinal and transverse directions which induce stresses in the plate linearly distributed across the thickness as shown with the stresses on the top surface both being compressive. Note that the moment m_x causes a direct stress in the x direction (σ_x).

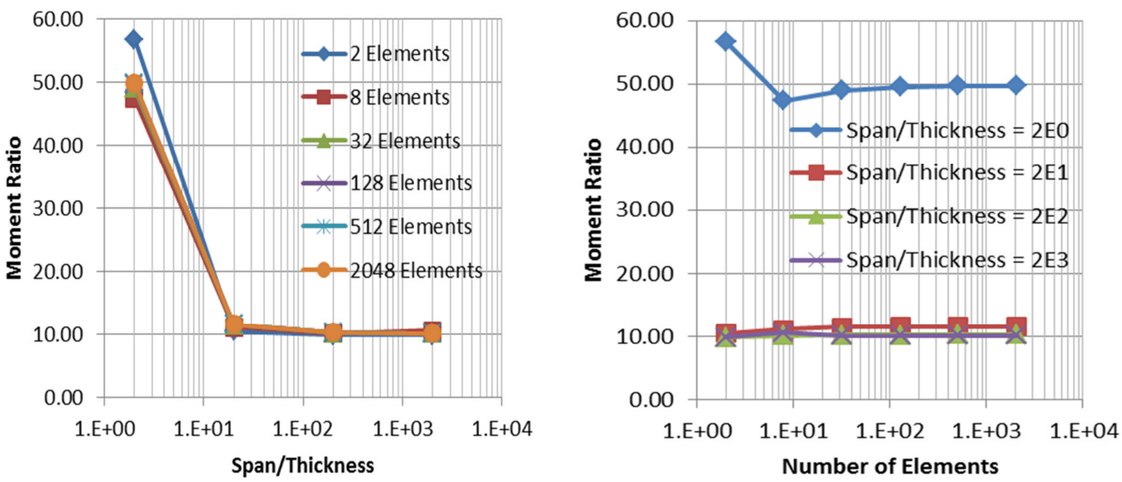
Figure 2: Moments and stresses at centre of Challenge Plate

I perused the manual for the finite element software and discovered it has no plate elements. The absence of plate elements is not uncommon in commercial systems since software vendors tend to assume that whenever an engineer wants to study plate-bending the problem might also include plate-membrane effects and so offers the user a choice of shell elements which include both plate-bending and plate-membrane effects simultaneously. I guess this is rather like the beam element which is the one-dimensional analogue of the shell element and also generally includes bending and axial effects simultaneously. Interestingly the options for the shell element enable me to switch off the bending part of the element but not the membrane part so it seems, even though my student's problem only involves plate-bending, that I shall have to, somewhat wastefully in my view, use the element with the membrane part switched on even though it will not be invoked in my analysis.

I am aware that my student's plate has a very large span/thickness ratio – the span used was 2m and the thickness of the plate was 10mm so the span/thickness ratio is 200. I understand that there are essentially two formulations for plate elements these being Reissner-Mindlin for 'thick' plates and Kirchhoff for 'thin' plates and I understand that the transition between thick and thin plates occurs at a span/thickness ratio of about ten. Thus, clearly, my student's plate is definitely in the thin region where Kirchhoff formulation is appropriate. The software manual offers two shell elements, a lower-order four-noded element and a higher-order eight-noded element both of which are quoted as being suitable for thin to moderately thick shell structures so should be ideal for my problem. The eight-noded element seems the best element to use; I always prefer to use higher-order elements since I will need fewer elements than a lower-order equivalent element to obtain decent results.

I will start off by modelling my student's plate using meshes of eight-noded elements and as I am curious to confirm the received wisdom regarding span/thickness ratios I will explore how the moment ratio at the centre of the plate converges with mesh refinement and for different span/thickness ratios. I will use the actual plate dimensions (2m x 1m) but I shall utilise symmetry and model just the top right hand quadrant. Now, do I need to worry about the value of Young's modulus? I don't think so because the plate problem is driven by a force (a uniformly distributed load) and the material is isotropic so the stiffness of the material, whilst affecting the displacements, will not influence the moments. Anyway I shall use a typical value for steel so that the deformations are sensible.

The results from my convergence study are summarised in figure 3. I started with a mesh of 1x2=2 elements and performed uniform mesh refinement with the final mesh comprising 2048 elements and I investigated span/thickness ratios between 2 and 2000 with a Poisson's ratio of 0.3.



	Number of Elements					
Span/Thickness	**2**	**8**	**32**	**128**	**512**	**2048**
2.E+00	56.74	47.36	48.97	49.53	49.68	49.72
2.E+01	10.52	11.17	11.48	11.58	11.61	11.61
2.E+02	9.86	10.15	10.23	10.28	10.30	10.30
2.E+03	9.84	10.61	10.18	10.18	10.19	10.19

Figure 3: Convergence of moment ratio with span/thickness ratio and mesh refinement

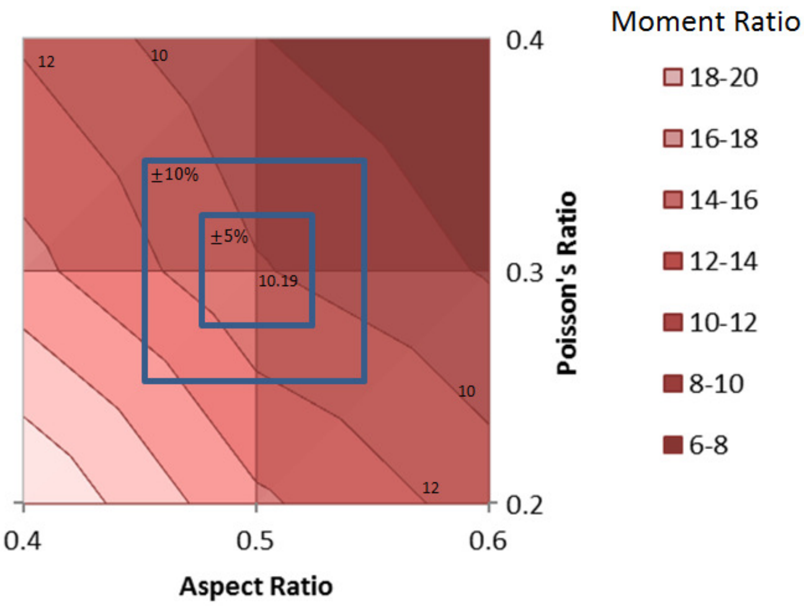
My results are rather satisfying in that convergence with both mesh refinement and span/thickness is observed. The suggested thick to thin plate transition does indeed seem to occur at a span/thickness ratio of ten (20 in this case because my results don't include the value of ten) and convergence with mesh refinement is clearly seen. From the results I have generated I am happy that for my student's plate configuration the results are independent of span/thickness ratio and that I now know the sort of mesh required to achieve accurate results.

It would be interesting now to see how sensitive the moment ratio is to the following parameters:

- Poisson's Ratio
- Aspect Ratio
- Positioning of Strain Gauge

Let me begin by considering sensitivity due to the first two parameters. I will look at aspect ratios in the range 0.4 to 0.6 and Poisson's ratio values in the range 0.2 to 0.4 and for each parameter I will generate results at 0.1 intervals.

The results from my sensitivity study are shown in figure 4. Lines of constant moment ratio are shown and two central boxes which mark the boundary for a 5% and 10% error in the two parameters.



Poisson's Ratio	Aspect Ratio 0.4	Aspect Ratio 0.5	Aspect Ratio 0.6
0.2	20.58	14.39	11.12
0.3	14.68	10.19	7.82
0.4	11.74	8.10	6.17

Figure 4: Moment ratio sensitivity (FE)

Looking at where the contour line corresponding to a moment ratio of 12 (the value from Timoshenko) crosses the error boxes indicates that this can only be achieved with an error in either the aspect ratio and/or Poisson's ratio in excess of 5%. My student's experimental setup used knife edge supports to represent the simple supports for this problem and there is no reason to suspect that she didn't position these correctly – but I will check with her!

I am still no nearer solving the problem of the difference in results although I am beginning to suspect that there might be an error in Timoshenko's text; I'd be surprised at this since this text is such a well revered and referenced text on the subject of plates that if there was an error then surely it would have been reported and updated. I am going to have to go back to the equations in Timoshenko's text and see if I can check the moment values he has produced in his table. Unfortunately he does not provide expressions for the moments but he does provide a series solution for the displacement from which the moments can be derived. As such I wrote a small program to evaluate the moments for the challenge plate.

I have compared the differences between Timoshenko's solution (generated by my program) and the finite element solution in table 1. The difference between the results are small (maximum of 0.33%) with the FE results being slightly greater than the series solution.

	Aspect Ratio		
Poisson's Ratio	**0.4**	**0.5**	**0.6**
0.2	-0.29	-0.21	-0.11
0.3	-0.27	-0.19	-0.10
0.4	-0.33	-0.21	-0.21

Table 1: Percentage difference in moment ratio {100 x (Program-FE)/Program}

The good agreement shown in table 1 implies that it is the results shown in Timoshenko's table (see figure 1) that are incorrect. I can use my program to check the two moment components listed in Timoshenko's table and this has been done in table 2.

	$x = a/2\,, y = 0$		
	$M_x = \beta_1 q a^2$	$M_y = \beta_1' q a^2$	**Moment Ratio**
$b/a = 0.5$	β_1	β_1'	M_x/M_y
Text	0.1235	0.0102	12.108
Program	0.1237	0.0122	10.168

Table 2: Moments and moment ratios from Timoshenko's text and the program

It is now clear that there is a significant difference in the values of M_y. There is likely to be a difference between the values presented in Timoshenko's text and those produced by modern software since in the absence of computers Timoshenko would have had to rely on tables of hyperbolic trigonometrical functions and rigorous checking procedures to ensure accuracy. Looking at the two numbers, however, I believe that the error is simply a typographical one in that if I transposes the fourth and fifth digits of the result presented by Timoshenko (0.0102) then I obtain 0.0120 which is much closer to the value obtained by the program and brings the worst percentage difference down to an acceptable value consistent with the other values presented.

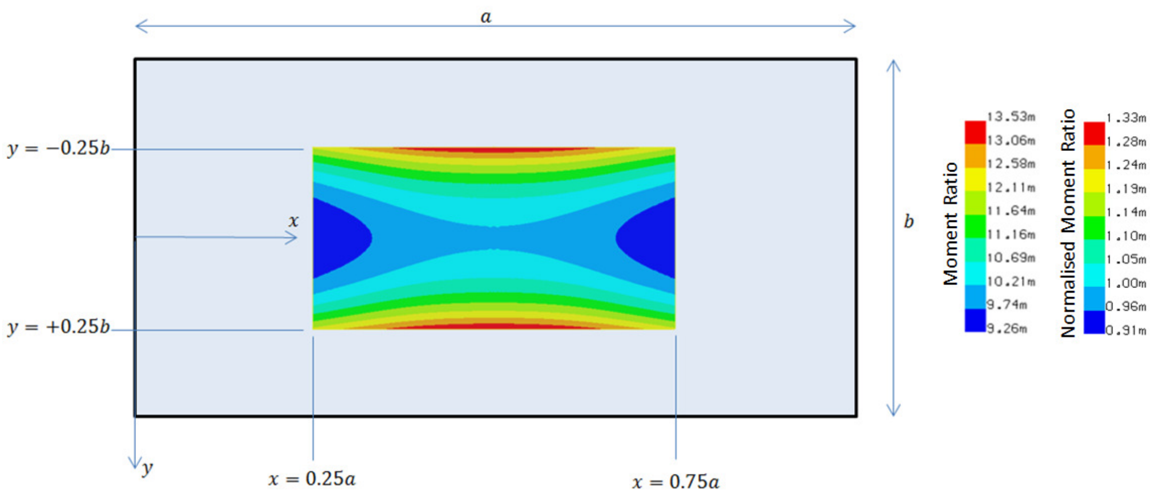


Figure 5: Moment ratio in central region of the challenge plate

I don't want to solve my student's problem for her and she would learn much by making the discovery herself. If I give her my program she could use it to discover the error for herself. I could put her off the scent a little by suggesting that she use the program to see how sensitive the moment ratio is to the location of the strain gauges. I was rather impressed when she came back with the diagram shown in figure 5. She explained that the moment ratio is rather insensitive to small deviations in the location of the strain gauges and then said that if my program was correct then there was an error in Timoshenko's text. A great result!!

Closure

This challenge has demonstrated that even in some of the most revered texts errors can occur; in this case almost certainly the error is a typographical one. The practising engineer needs to be aware of this and where possible take the opportunity to verify the numbers with which he/she is presented. As demonstrated in this challenge, the finite element method is an ideal tool for the engineer to use in this process of verification.

Having discovered an error the question now is what should the supervisor do? Well, the error in M_y at the centre of the challenge plate is about 20% and Timoshenko's result is lower than the correct value, i.e. it is a non-conservative prediction of the actual moment. If the engineer is using Timoshenko's data for the elastic design of a steel plate then he/she will spot that the highest stress occurs at point B rather than point A (see figure 1). As there is no error in the moments at point B then the design is not going to be compromised by this error. However, if the engineer is using Timoshenko's data to specify the reinforcement in a reinforced concrete slab and if he/she is looking to optimise the reinforcement then he/she may underestimate the transverse reinforcement requirement at the centre of the slab potentially leading to the possibility of cracking and the necessity to rely on plastic moment redistribution in order to achieve the required strength.

The supervisor felt, as a matter of professional integrity, that he should let the publishers of Timoshenko's text know about the error. It appeared, though, that as no reprints of this text were scheduled it would remain uncorrected. As such, the supervisor prepared a short article outlining the error he had discovered and it was published both in full in the UK [3] and in a shorter version in the US [4].

References

[1] S.P. Timoshenko & S. Woinowsky-Krieger, 'Theory of Plates and Shells', 2nd Edition, McGraw-Hill International Series, 28th Printing 1989. ISBN 0-07-Y85820-9.

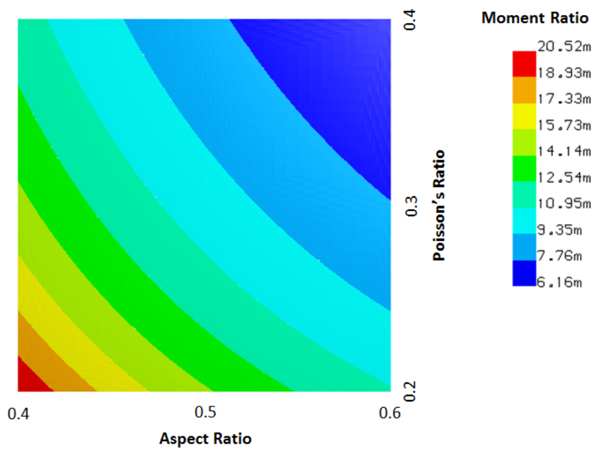
[2] T. Hellen, 'How to use Beam, Plate and Shell Elements', NAFEMS 'How to …' series, Number 34, 2006.

[3] A.C.A. Ramsay & E.A.W. Maunder, 'An Error in Timoshenko's "Theory of Plates and Shells"', NAFEMS Benchmark Magazine, January 2016.

[4] A.C.A. Ramsay & E.A.W. Maunder, 'An Error in Timoshenko's "Theory of Plates and Shells"', Structure Magazine, March 2016.

Appendix 1: Using the Program to Confirm FE Sensitivity Study

The program was used to repeat the rather coarse sensitivity study conducted using the FE model and the results are shown in figure 6.



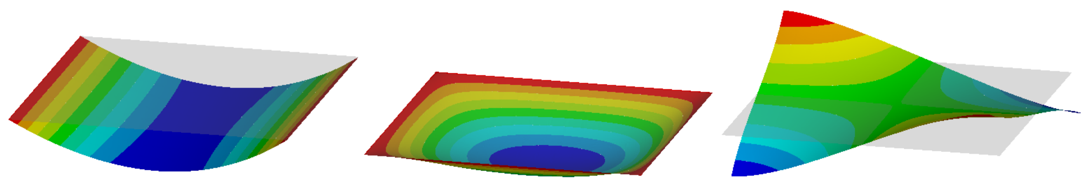
Poisson's Ratio	Aspect Ratio 0.4	0.5	0.6
0.2	20.52	14.36	11.11
0.3	14.64	10.17	7.81
0.4	11.70	8.08	6.16

Figure 6: Moment ratio sensitivity study (Program)

The results in figure 6 compare well with those in figure 4 and confirm the conclusions made from the FE sensitivity study.

Appendix 2: Moment Fields from Software

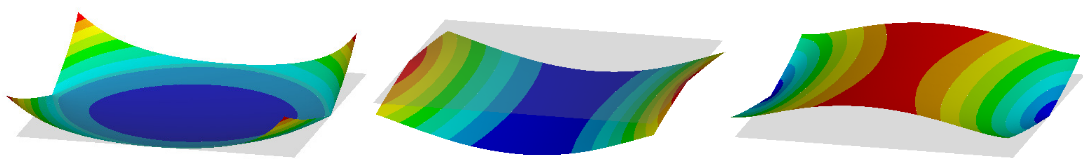
Having developed software to evaluate the moments for the challenge plate it is possible to plot them out in the form of contour plots and it is instructive so to do. The Cartesian moment fields, with units Nm/m, are shown in figure 7 and correspond to a 1Pa UDL. Note that as this is a Kirchhoff solution there is no requirement for the torsional moments to be zero along the boundary.



$-0.51 \le M_x \le 0$ $\quad$ $-0.05 \le M_y \le 0$ $\quad$ $-0.14 \le M_{xy} \le +0.14$

Figure 7: Cartesian Moments from Program (nu=0.3)

The moment invariants (functions of the moments that are invariant with the coordinate system such as the principal moments and the von Mises moments) are plotted in figure 8 and the von Mises moment field shows that the point of first yield is at the centre of the free edges as expected.



$-0.05 \le M_1 \le +0.14$ $\quad$ $-0.51 \le M_2 \le 0$ $\quad$ $0 \le M_{vM} \le +0.51$

Figure 8: Moment invariants from program (nu=0.3)

The principal moment trajectories are plotted in figure 9 and show both the magnitude (colour and thickness) and direction of the principal moments and provide, in a single figure, a complete picture of the statics.

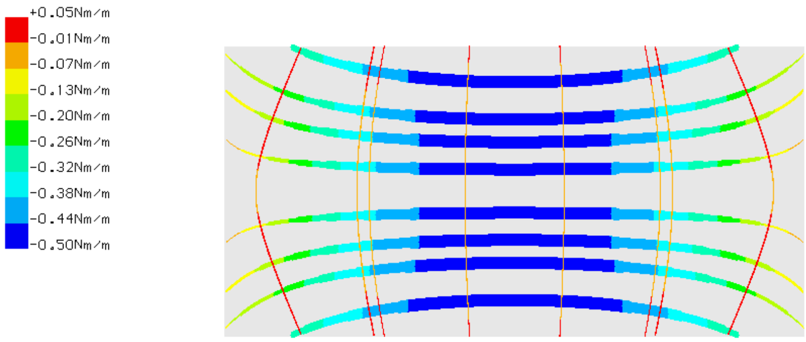


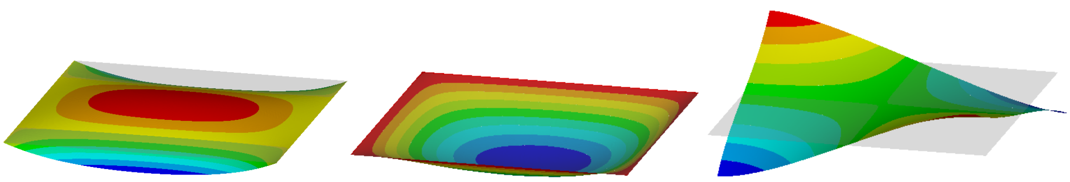
Figure 9: Principal moment trajectories from program (nu=0.3)

A further simple check on the distributions can be made by setting Poisson's ratio to zero. This has the effect of uncoupling the moment fields such that the only remaining non-zero moment field is M_x which now becomes constant in the y direction – see figure 10 – and is identical to the solution that would have been achieved if the plate had been idealised as a narrow beam.

$-0.5 \leq M_x \leq 0$ $\quad M_y \quad M_{xy}$

Figure 10: Cartesian moments from program (nu=0)

Given two solutions for the same problem but with different Poisson's ratio values enables us to conjure up a new moment field as the difference between that achieved with nu=0.3 and nu=0. This field is shown in figure 11.



$-0.01 \leq M_x \leq +0.01 \quad -0.05 \leq M_y \leq 0 \quad -0.14 \leq M_{xy} \leq +0.14$

Figure 11: Cartesian moments from program (hyperstatic)

The moment field shown in figure 11 has the interesting property that it satisfies the equations of equilibrium with no applied load. In other words it is a hyperstatic moment field in the sense of Kirchhoff plate theory.

For completeness the principal moment trajectories for the case of nu=0 and the hyperstatic moment field are shown in figures 12 and 13.

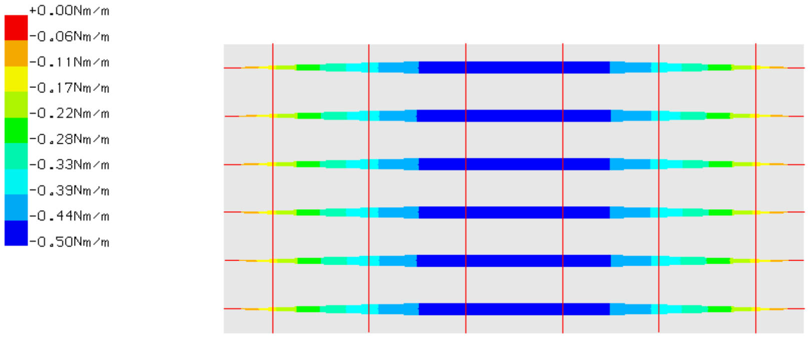


Figure 12: Principal moment trajectories from program (nu=0)

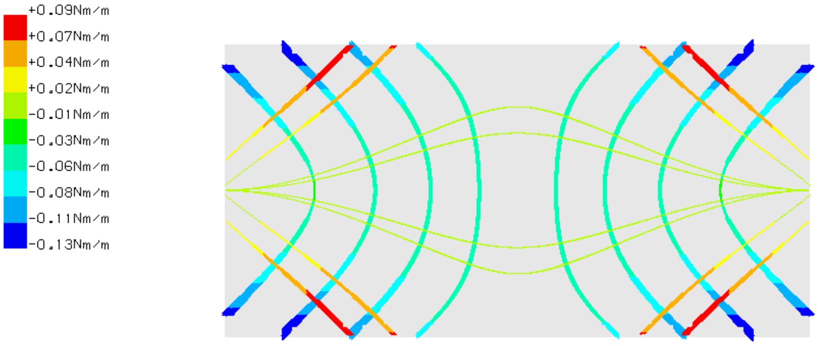


Figure 13: Principal moment trajectories from program (hyperstatic)

In the context of, for example, pressure vessel design, a hyperstatic field would be termed a secondary moment field with the corresponding primary moment field being that obtained for nu=0. It should be noted, however, that different primary and secondary moment fields would have been obtained if different values of Poisson's ratio had been chosen and this illustrates the non-uniqueness of this approach for separating a moment field into primary and secondary fields.

In the context of moment fields then a lower-bound limit analysis might be understood as starting off with a particular moment field, which equilibrates the applied loading, and then adding a set of hyperstatic fields (the number of these being equal to the degree of statical indeterminacy of the structure) with variable amplitudes. The amplitudes are then used as variables in a mathematical programme to maximise the load factor whilst ensuring that the moment field does not violate the yield criterion.

It is worth pointing out that whilst the elastic moment field is unique, the moment field at collapse (in a limit analysis) is not although the load factor is unique. In otherwords there are a range of moment fields that would produce the same collapse load factor. In this manner it is simple to see why the collapse load is not influenced by the presence of hyperstatic fields in a structure due to, for example, residual stresses and/or thermal gradients.

Appendix 3: Supervisor's Notes on Refinement Strategies

Most commercial finite element software has what are often described as lower-order and higher-order elements for most of the usual structural forms, e.g. beams, shells, solids etc. A few of these codes go further than this including what are described as *p*-type elements. These elements have variable polynomial degree (constant, linear, quadratic etc.) and the engineer can then adopt a *p*-type refinement strategy (increasing the degree of the element) in addition to the more usual *h*-type refinement strategy where more elements of the same degree are used.

When faced with a choice of lower or higher-order elements the engineer might well, unless there is any reason not so to do, opt for the higher-order element since he/she might be aware that these are likely to lead, mesh for mesh, to a more accurate solution. This idea is of course true and there are significant benefits in the economy of solutions (computational effort required) obtained using a *p*-type approach over an *h*-type refinement strategy.

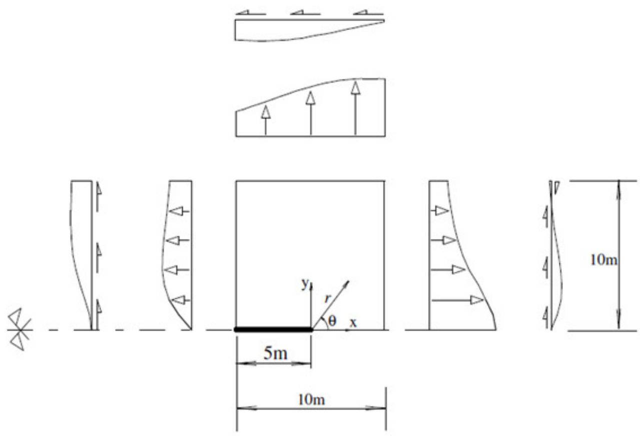
This idea can be simply demonstrated by considering a practical engineering problem and running it through a finite element code that has *p*-type elements. The problem considered is that of a rectangular plate with a crack parallel to the edges of the plate. This is a plate membrane problem and it will be assumed that the plate is thin so that a plane stress constitutive relationship is appropriate. There is a closed-form solution for this problem which is shown in figure 14 along with the boundary tractions that need to be applied to the model.

This problem has infinite stress at the origin (where r = zero) and, clearly, this stress will never be recovered with elements using polynomial fields however high a degree is used. However, this is a perfectly reasonable problem to consider as such singularities in stress (maybe due to geometric features such as sharp re-entrant corners or other features like the application of a point load) are often present in practical problems considered by finite element analyses.

$$\sigma_x = \frac{100}{\sqrt{r}}\cos\frac{\theta}{2}\{1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\}$$

$$\sigma_y = \frac{100}{\sqrt{r}}\cos\frac{\theta}{2}\{1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\}$$

$$\tau_{xy} = \frac{100}{\sqrt{r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$



(a) Stress Field (b) Boundary Tractions

Figure 14: Plate membrane problem with crack

Rather than look at the convergence of, for example, the maximum stress which in this case is infinite, one can instead look at the convergence of the strain energy which, being an integral quantity, has a finite value that can be evaluated for the stress field given in figure 14.

Since the strain energy for the stress field in figure 14 is known, it is possible to evaluate the error in strain energy for a given finite element model; let us define the error in strain energy as the difference between the finite element value and the theoretical value. Using a series of four uniformly refined meshes and an element with polynomial degree in the range $1 \le p \le 10$ the error in strain energy has been evaluated and the values plotted in figure 15.

The convergence of the finite element strain energy with degrees of freedom shown in figure 15 includes some annotation that is intended to aid understanding. Four meshes are considered. These meshes adopt a uniform refinement process where the elements of the previous mesh are subdivided leading to a reduction in *h*, a characteristic dimension of an element, by a factor of one half. Lines showing the convergence of the error in strain energy resulting from *p*-type and h-type refinement can be added by joining points of equal *p* and *h* respectively. Two such lines have been plotted showing *p*-type refinement for the first mesh (*h*=1) and *h*-type refinement for the linear element with *p*=1. The first point to note is the gradient of these two lines; the *p*-type refinement line has a greater gradient than the corresponding *h*-type line demonstrating that, with respect to the number of degrees of freedom, *p*-type refinement leads to a greater rate of convergence.

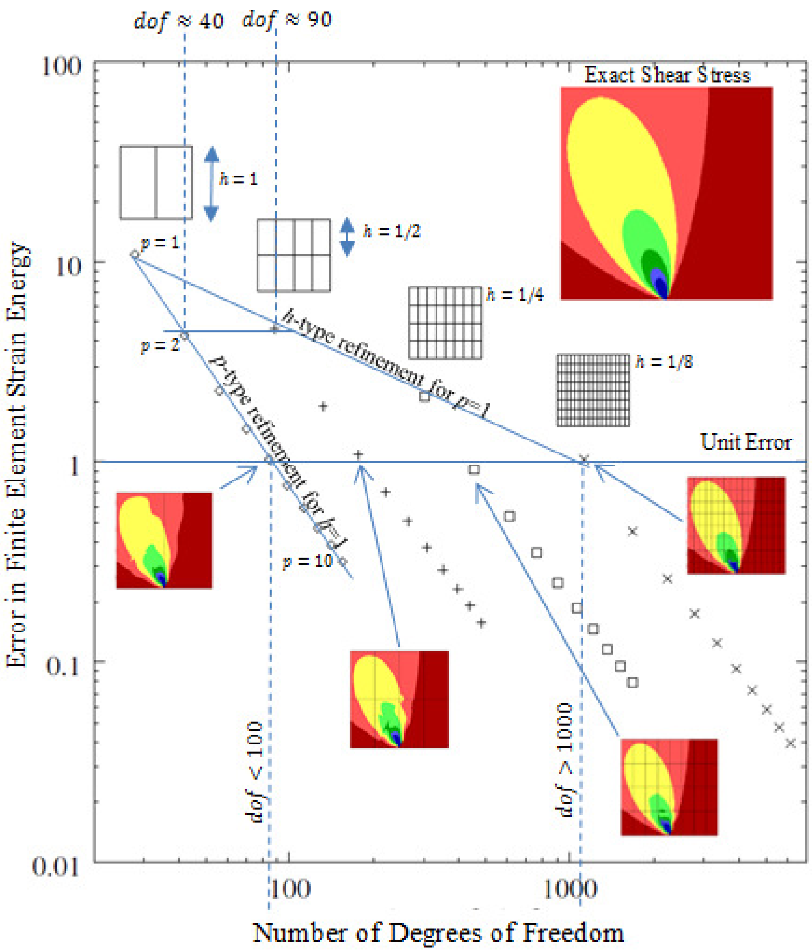


Figure 15: Error in strain energy as a function of number of degrees of freedom

A horizontal line has been added to the figure indicating a unit error in the finite element strain energy (the theoretical strain energy is 62.4Nm so the percentage error in strain energy is about 1.6%). Vertical lines have also been added where the *p*-type and *h*-type lines intersect the horizontal lines. These vertical lines show that for the same error (unit value in this case) the number of degrees of freedom required for the *p*-type refinement case is one order of magnitude less than that required for *h*-type refinement. Since the 'computational effort' (this is not a precise term but one that might be understood to be proportional to the elapsed time for solving a problem) required to solve a finite element problem increases at a greater than linear rate it is clear that there is a distinct advantage in terms of computational effort to adopt a *p*-type refinement strategy (if the computational effort was quadratic with number of degrees of freedom then an increase of one magnitude in the number of degrees of freedom translates to two orders of magnitude increase in computational effort). Contour plots of the finite element shear stress have been added to the figure for the four meshes with *p* values that give close to unit error and these may be compared, visually, with the theoretical shear stress shown in the top right-hand corner.

The significant reduction in computational effort noted above when using *p*-type refinement requires the availability of high degree elements. However even if only the standard lower-order and higher-order elements are available to the engineer then a similar although not as dramatic reduction in computational effort can be achieved. In the top left-hand corner of the figure the number of degrees of freedoms required to capture the solution with a similar error for the p=1, h=1/2 and p=2, h=1 meshes (similar to the lower-order and higher-order elements found in most commercial systems) are shown. If, again, we make the assumption that the computational effort is in proportion to the square of the number of degrees of freedom there is still a factor of just over five between the two refinement approaches which is certainly something to be considered by the engineer particularly when his/her model is large.

Thus the finding for the practising engineer using a commercial finite element software package is that unless there is any reason not to, then the use of higher-order elements will enable him/her to obtain a given accuracy in a more efficient manner than would be the case if lower-order elements had been used.